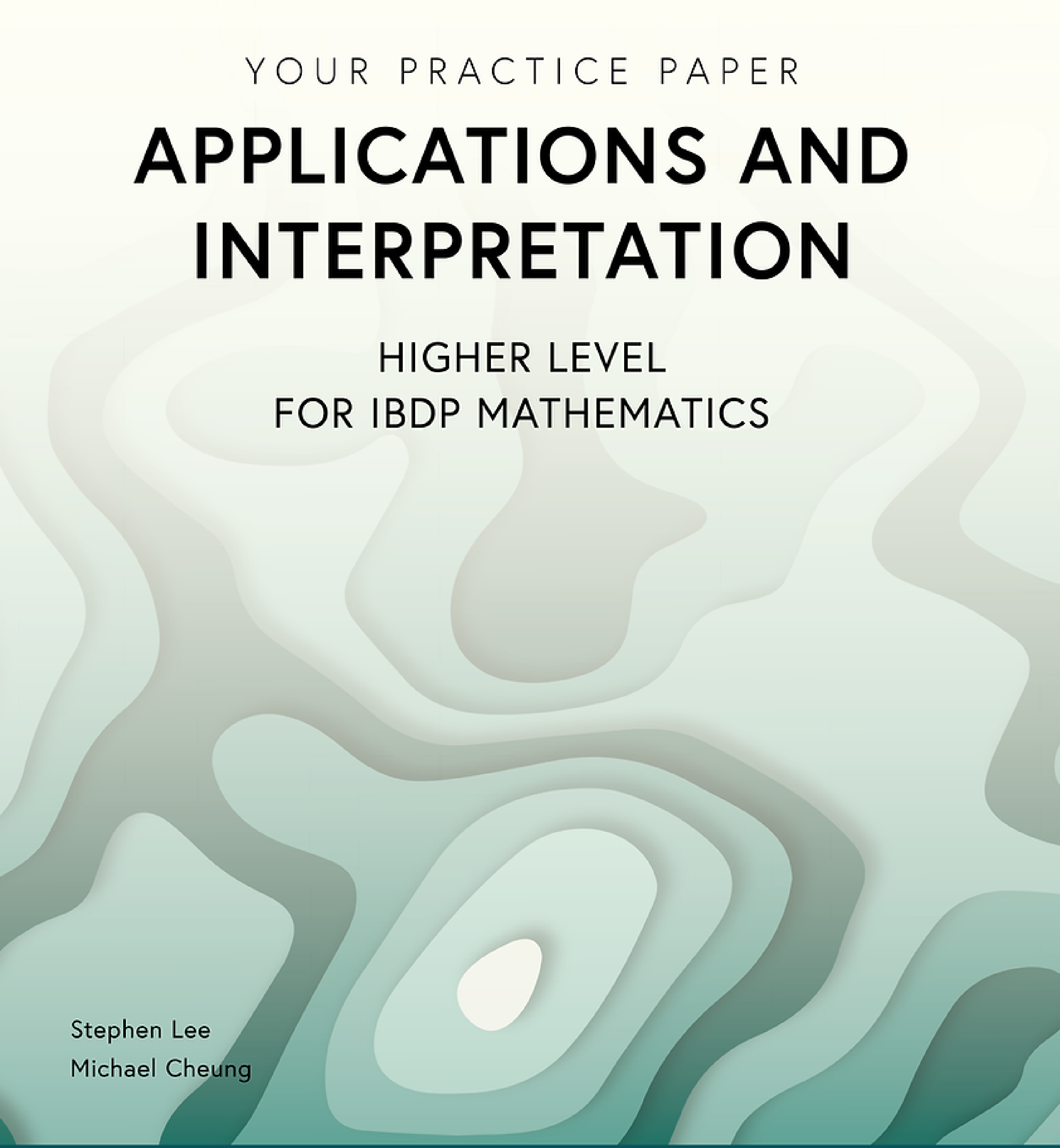
YOUR PRACTICE PAPER

APPLICATIONS AND INTERPRETATION

HIGHER LEVEL
FOR IBDP MATHEMATICS



Stephen Lee
Michael Cheung

- 4 Sets of Practice Papers
- Distributions of Exam Topics
- Exam Format Analysis
- Comprehensive Formula List

SE PRODUCTION LIMITED

Booklet 0.1

Your Practice Paper
Applications and Interpretation Higher Level for IBDP Mathematics

Stephen Lee
Michael Cheung

SE Production Limited

Your Practice Paper
Applications and Interpretation
Higher Level
for IBDP Mathematics

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Book cover: Mr. M. H. Lee

Contents

The solution page of this book

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Mr. Stephen Lee received his Bachelor of Science in Mathematics and Statistics, and Master of Statistics from The University of Hong Kong. During his postgraduate studies at HKU, he was a teaching assistant in the Department of Statistics and Actuarial Science, The University of Hong Kong, where he conducted tutorial lessons for undergraduate students. Later on, he received the Postgraduate Diploma of Education in Mathematics from the Chinese University of Hong Kong. He is currently a frontline teacher in an IB World School. Apart from local syllabus in Hong Kong, he has experience in teaching various levels in IBDP Mathematics. He is also an examiner of the International Baccalaureate Organization (IBO). Furthermore, he is also the chief author of the book series: **Your Personal Coach Series – HKDSE Mathematics (Compulsory Part) Conventional Questions and Multiple Choice Questions**, **Your Practice Set – Analysis and Approaches for IBDP Mathematics Book 1 & 2**, **Your Practice Set – Applications and Interpretation for IBDP Mathematics Book 1 & 2**, **Your Practice Paper – Analysis and Approaches Standard Level for IBDP Mathematics**, **Your Practice Paper – Analysis and Approaches Higher Level for IBDP Mathematics** and **Your Practice Paper – Applications and Interpretation Standard Level for IBDP Mathematics**.

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Mr. Michael Cheung has a strong Mathematics background and has been teaching Mathematics for more than 10 years. He conducted tutorial classes in fluent English to international students from different international schools. He has been teaching Mathematics in an IB world school. As an IB examiner, he needs to help on marking the IB exam papers every year. Based on his experience, he is very familiar with IB syllabus and knows about different question styles in real exam.

Foreword

People in this world have different views on academic success. Some people think that academic success is measured by scores on examinations, while some may think that it should be measured by the happiness in learning. From our point of view, academic success is that students can learn in an effective way and have enjoyment in the learning process. Students can find learning interesting and have motivation if the learning process is effective, and thus learning becomes enjoyable and the chance of getting good academic results will be greater.

In preparing this book, our team was guided by our experience and interest in teaching IBDP Mathematics. This book is designed to help students to have a final preparation in the brand new challenging two-year International Baccalaureate Diploma Program. This book helps students to review all important concepts in Applications and Interpretation Higher Level, and helps students to understand the formats of assessment-styled questions. No doubt, this book can help you to achieve high exam scores in IBDP Mathematics. By going through this book, you will find that the questions can help you to answer the structured questions confidently.

To sum up, this book is not only to be a successful practice source, but also to serve as valuable resource for students of each area.

SE Production Team

Updates

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for any updates on this book.

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Practice Paper Analysis

	Paper 1	Paper 2	Paper 3
Full Mark	110	110	55
Time	120 Minutes	120 Minutes	60 Minutes
Calculator	Needed		
Format	18 Short Questions	7 Structured Questions	2 Structured Questions

Categories	Topics	Mark Ranges	Percentages
Category 1: Algebra	Standard Form	31 to 36 Marks	14% to 16%
	Approximation and Error		
	Systems of Equations		
	Arithmetic Sequences		
	Geometric Sequences		
	Financial Mathematics		
	Complex Numbers		
	Matrices		
Category 2: Functions	Functions	33 to 39 Marks	15% to 18%
	Quadratic Functions		
	Exp. and Log. Functions		
	Coordinate Geometry		
Category 3: Geometry	Voronoi Diagrams	49 to 53 Marks	22% to 24%
	Trigonometry		
	2-D Trigonometry		
	Areas and Volumes		
	Vectors & Graph Theory		
Category 4: Statistics	Statistics & Probability	51 to 54 Marks	23% to 25%
	Discrete Distributions		
	Binomial Distribution		
	Poisson Distribution		
	Normal Distribution		
	RV Linear Combinations		
	Point Estimation		
	Interval Estimation		
	Bivariate Analysis		
	Statistical Tests		
Category 5: Calculus	Differentiation	43 to 50 Marks	20% to 23%
	Integration & Trap. Rule		
	Differential Equations		

Distributions of Questions		Set 1		Set 2		Set 3		Set 4	
		P1	P2	P1	P2	P1	P2	P1	P2
1	Standard Form								
	Approximation and Error								
	Systems of Equations								
	Arithmetic Sequences	2				1			
	Geometric Sequences			3				3	
	Financial Mathematics	7		5		7		6	
	Complex Numbers	15		16			4		6
	Matrices		5		7	17		16	
2	Functions	10		15		6,13		11	
	Quadratic Functions	8		6				7	
	Exp. and Log. Functions	16		10		11	2	15	
	Coordinate Geometry		1		1				1
3	Voronoi Diagrams	4		4		5		4	
	Trigonometry	14		2		18			
	2-D Trigonometry	3		8		3		8	
	Areas and Volumes	6				8		1	
	Vectors		7	11	5		6		4
	Graph Theory		4		6		5	13	7
4	Statistics	1		1		2		2	
	Probability							18	
	Discrete Distributions					4			
	Binomial Distribution	5						5	
	Poisson Distribution	11		9		14			
	Normal Distribution			7					2
	RV Linear Combinations	9		12					
	Point Estimation			13					
	Interval Estimation	18				12		10	
	Bivariate Analysis		2	17		16	1	12	
	Statistical Tests	13			2	10		14	
5	Differentiation	12	3	14	3	9		17	
	Integration & Trap. Rule	17			4	15	3		3
	Differential Equations		6	18			7	9	5

Ways to Use This Book

[2]	Number of marks for a question
M1	A mark is assigned when the corresponding method is clearly shown
(M1)	A mark is assigned when the corresponding method is not clearly shown but is shown in the following correct working
A1	A mark is assigned when the correct answer is clearly shown
(A1)	A mark is assigned when the correct answer is not clearly shown but is shown in the following correct working
R1	A mark is assigned when the reasoning statement is clearly shown
AG	No mark is assigned as the final step (usually would be answer) is already given from the question

GDC Skills

Some implicit skills of TI-84 Plus CE that you might not hear before

Scenario 1: Solving $f(x)=g(x)$ in Functions

Step 1: Set $f(x)-g(x)=0$

Step 2: Input $\mathrm{Y}_1=f(x)-g(x)$ in the graph function

Step 3: Set the screen size from $\boxed{\text{window}}$

- ✓ $x\min$ and $x\max$: You can refer to the domain given in the question
- ✓ $y\min$ and $y\max$: You can set $y\min=-1$ and $y\max=1$ if you wish to find the x-intercept only

Scenario 2: Finding the number of years, n, when $f(x)=g(x)$ is in the exponent of an exponential model, in Arithmetic Sequences / Geometric Sequences / Logarithmic Functions

Step 1: Set the right-hand-side of the expression to be zero

Step 2: Input $\mathrm{Y}_1=$the left-hand-side of the expression in the graph function

Step 3: Set the screen size from $\boxed{\text{window}}$

- ✓ $x\min$: You can set $x\min=0$ as n represents the number of years which must be a positive integer

Scenario 3: Finding the x-intercept from the window

- ✓ Assume that the domain is $0\le x\le 100$, and it is clearly shown that the curve cuts the x-axis once only on the left part of the screen
- ✓ You can set the left bound and the right bound to be 0 and 50 respectively to find the x-intercept efficiently, as 50 is the midpoint of the x-axis

Scenario 4: Finding an area under a curve and above the x-axis

✓ Apart from using the function $\boxed{\text{MATH}}$ $\boxed{9}$, you can sketch the curve and use the function $\boxed{\text{2nd}}$ $\boxed{\text{trace}}$ $\boxed{7}$, and then set the lower limits and the upper limits.

Scenario 5: Finding probabilities in a Binomial distribution, in the form $\mathrm{P}(X < c)$, $\mathrm{P}(X > c)$ or $\mathrm{P}(X \geq c)$

✓ You need to change the probability to the form $\mathrm{P}(X \leq C)$, and then use the function $\boxed{\text{2nd}}$ $\boxed{\text{vars}}$ $\boxed{\text{B}}$ to choose $\mathrm{binomcdf}$.

Scenario 6: Finding an unknown quantity from the TVM Solver

$$\boxed{\begin{array}{l} \mathrm{N}=5 \\ \mathrm{I\%}=6 \\ \mathrm{PV}=-24000 \\ \mathrm{PMT}=0 \\ \mathrm{FV}=? \\ \mathrm{P/Y}=1 \\ \mathrm{C/Y}=1 \\ \mathrm{PMT:END} \end{array}} \rightarrow \boxed{\begin{array}{l} \mathrm{N}=5 \\ \mathrm{I\%}=6 \\ \mathrm{PV}=-24000 \\ \mathrm{PMT}=0 \\ \mathrm{FV}=0 \\ \mathrm{P/Y}=1 \\ \mathrm{C/Y}=1 \\ \mathrm{PMT:END} \end{array}}$$

✓ You can set the unknown quantity to be zero in order to execute the program. In the above example, the future value of a compound interest problem is going to be found. You can set FV to be zero and then choose $\mathrm{tvm\_FV}$ to calculate the future value.

Booklet 0.2

Formula List of
Applications and Interpretation
Higher Level
for IBDP Mathematics

Analysis & Approaches Standard Level	Analysis & Approaches Higher Level
Applications & Interpretation Standard Level	Applications & Interpretation Higher Level

1 Standard Form

✓ Standard Form:

A number in the form $(\pm)a\times10^k$, where $1\le a<10$ and k is an integer

2 Approximation and Error

✓ Summary of rounding methods:

2.71828	Correct to 3 significant figures	Correct to 3 decimal places
Round off	2.7**2**	2.71**8**

✓ Consider a quantity measured as Q and correct to the nearest unit d:

$\frac{1}{2}d$: Maximum absolute error

$Q-\frac{1}{2}d\le A<Q+\frac{1}{2}d$: Range of the actual value A

$Q-\frac{1}{2}d$: Lower bound (Least possible value) of A

$Q+\frac{1}{2}d$: Upper bound of A

$\frac{\text{Maximum absolute error}}{Q}\times100\%$: Percentage error

3 Functions

✓ The function $y = f(x)$:

1. $f(a)$: Functional value when $x = a$
2. Domain: Set of values of x
3. Range: Set of values of y

✓ Properties of rational function $y = \frac{ax+b}{cx+d}$:

1. $y = \frac{1}{x}$: Reciprocal function
2. $y = \frac{a}{c}$: Horizontal asymptote
3. $x = -\frac{d}{c}$: Vertical asymptote

✓ $f \circ g(x) = f(g(x))$: Composite function when $g(x)$ is substituted into $f(x)$

✓ Steps of finding the inverse function $y = f^{-1}(x)$ of $f(x)$:

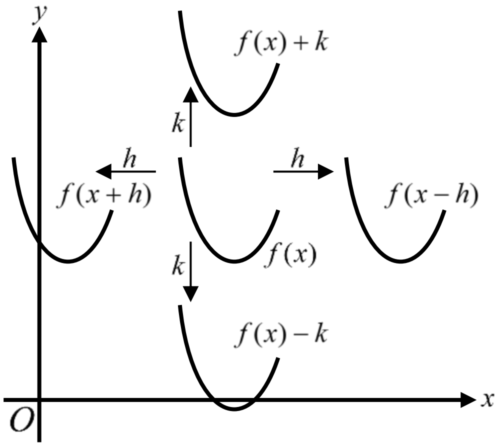
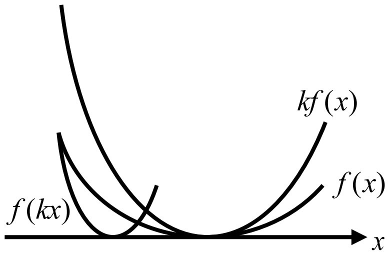
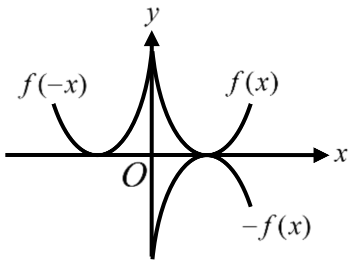
1. Start from expressing y in terms of x
2. Interchange x and y
3. Make y the subject in terms of x

✓ Properties of $y = f^{-1}(x)$:

1. $f(f^{-1}(x)) = f^{-1}(f(x)) = x$
2. The graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ about $y = x$

✓ $f^{-1}(x)$ exists only when $f(x)$ is one-to-one in the restricted domain

✓ Summary of transformations:

	$f(x)\to f(x)+k$: Translate upward by k units
	$f(x)\to f(x)-k$: Translate downward by k units
	$f(x)\to f(x+h)$: Translate to the left by h units
	$f(x)\to f(x-h)$: Translate to the right by h units
	$f(x)\to kf(x)$: Vertical stretch of scale factor k
	$f(x)\to f(kx)$: Horizontal compression of scale factor k
	$f(x)\to -f(x)$: Reflection about the x-axis
	$f(x)\to f(-x)$: Reflection about the y-axis

✓ Variations:

1. $y=kx, k\neq 0$: y is directly proportional to x
2. $y=\dfrac{k}{x}, k\neq 0$: y is inversely proportional to x

4 Quadratic Functions

✓ General form $y=ax^2+bx+c$, where $a\neq 0$:

$a>0$	The graph opens upward
$a<0$	The graph opens downward
c	y-intercept
$h=-\dfrac{b}{2a}$	x-coordinate of the vertex
$k=ah^2+bh+c$	y-coordinate of the vertex
	Extreme value of y
$x=h$	Equation of the axis of symmetry

✓ Other forms:

1. $y=a(x-h)^2+k$: Vertex form
2. $y=a(x-p)(x-q)$: Factored form with x-intercepts p and q

✓ $h=-\dfrac{b}{2a}=\dfrac{p+q}{2}$

✓ The x-intercepts of the quadratic function $y=ax^2+bx+c$ are the roots of the corresponding quadratic equation $ax^2+bx+c=0$

5 Exponential and Logarithmic Functions

✓ $y=a^x$: Exponential function, where $a\neq 1$

✓ $y=\log_a x$: Logarithmic function, where $a>0$

✓ $y=\log x=\log_{10} x$: Common Logarithmic function

✓ $y=\ln x=\log_e x$: Natural Logarithmic function, where $e=2.71828...$ is an exponential number

✓ Properties of the graphs of $y = a^x$:

$a > 1$	$0 < a < 1$
y-intercept $= 1$	
y increases as x increases	y decreases as x increases
y tends to zero as x tends to negative infinity	y tends to zero as x tends to positive infinity
Horizontal asymptote: $y = 0$	

✓ Laws of logarithm, where a, b, c, p, q, $x > 0$:

1. $x = a^y \Leftrightarrow y = \log_a x$
2. $\log_a 1 = 0$
3. $\log_a a = 1$
4. $\log_a p + \log_a q = \log_a pq$
5. $\log_a p - \log_a q = \log_a \dfrac{p}{q}$
6. $\log_a p^n = n \log_a p$
7. $\log_b a = \dfrac{\log_c a}{\log_c b}$

✓ $f(x) = \dfrac{L}{1 + Ce^{-kx}}$: Logistic function, where L, C and k are positive constants

✓ Semi-log model:

1. $y = k \cdot a^x \Leftrightarrow \ln y = (\ln a)x + \ln k$: Semi-log model
2. $\ln a$: Gradient of the straight line graph on $\ln y$-x plane
3. $\ln k$: Vertical intercept of the straight line graph on $\ln y$-x plane

✓ Semi-log and log-log models:

1. $y = k \cdot x^n \Leftrightarrow \ln y = n \ln x + \ln k$: Log-log model
2. n : Gradient of the straight line graph on $\ln y$-$\ln x$ plane
3. $\ln k$: Vertical intercept of the straight line graph on $\ln y$-$\ln x$ plane

6 Systems of Equations

✓ $\begin{cases} ax+by=c \\ dx+ey=f \end{cases}$: 2×2 system

✓ $\begin{cases} ax+by+cz=d \\ ex+fy+gz=h \\ ix+jy+kz=l \end{cases}$: 3×3 system

✓ The above systems can be solved by $\boxed{\text{PlySmlt2}}$ in TI-84 Plus CE

7 Arithmetic Sequences

✓ Properties of an arithmetic sequence u_n:

1. u_1: First term
2. $d=u_2-u_1=u_n-u_{n-1}$: Common difference
3. $u_n=u_1+(n-1)d$: General term (nth term)
4. $S_n=\frac{n}{2}\left[2u_1+(n-1)d\right]=\frac{n}{2}\left[u_1+u_n\right]$: The sum of the first n terms

✓ $\sum_{r=1}^{n}u_r=u_1+u_2+u_3+\cdots+u_{n-1}+u_n$: Summation sign

8 Geometric Sequences

✓ Properties of a geometric sequence u_n:

1. u_1: First term
2. $r=u_2\div u_1=u_n\div u_{n-1}$: Common ratio
3. $u_n=u_1\times r^{n-1}$: General term (nth term)
4. $S_n=\frac{u_1(1-r^n)}{1-r}$: The sum of the first n terms

✓ $S_\infty = \dfrac{u_1}{1-r}$: The sum to infinity of a geometric sequence u_n, given that $-1 < r < 1$

9 Financial Mathematics

✓ Compound Interest:

PV : Present value

$r\%$: Interest rate per annum (per year)

n: Number of years

k: Number of compounded periods in one year

$FV = PV\left(1+\dfrac{r}{100k}\right)^{kn}$: Future value

$I = FV - PV$: Interest

✓ Inflation:

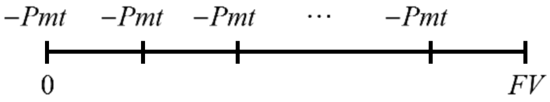
$i\%$: Inflation rate

$R\%$: Interest rate compounded yearly

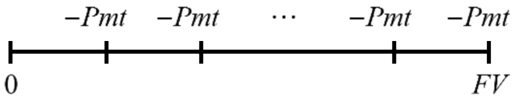
$(R-i)\%$: Real rate

✓ Annuity:

1. Payments at the beginning of each year

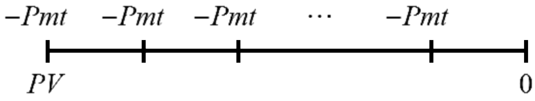


2. Payments at the end of each year

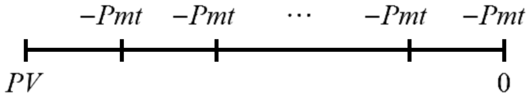


✓ Amortization:

1. Payments at the beginning of each year



2. Payments at the end of each year



10 Coordinate Geometry

✓ Consider the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ on a x - y plane:

1. $m = \frac{y_2 - y_1}{x_2 - x_1}$: Slope of PQ
2. $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$: Distance between P and Q
3. $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$: Mid-point of PQ

✓ Consider the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ on a x - y - z plane:

1. z -axis: The axis perpendicular to the x - y plane
2. $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$: Distance between P and Q
3. $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$: Mid-point of PQ

✓ Forms of straight lines with slope m and y -intercept c:

1. $y = mx + c$: Slope-intercept form
2. $Ax + By + C = 0$: General form

✓ Ways to find the x -intercept and the y -intercept of a line:

1. Substitute $y = 0$ and make x the subject to find the x -intercept
2. Substitute $x = 0$ and make y the subject to find the y -intercept

11 Voronoi Diagrams

✓ Elements in Voronoi Diagrams:

Site: A given point

Cell of a site: A collection of points which is closer to the site than other sites

Boundary: A line dividing the cells

Vertex: An intersection of boundaries

✓ Related problems:

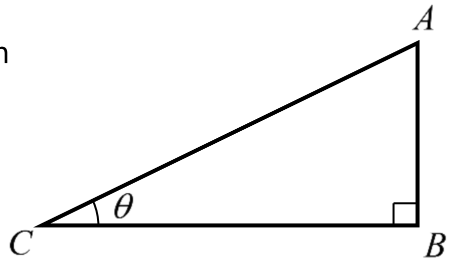
1. Nearest neighbor interpolation
2. Incremental algorithm
3. Toxic waste dump problem

12 Trigonometry

✓ Consider a right-angled triangle ABC:

$\mathrm{AB}^2+\mathrm{BC}^2=\mathrm{AC}^2$: Pythagoras' Theorem

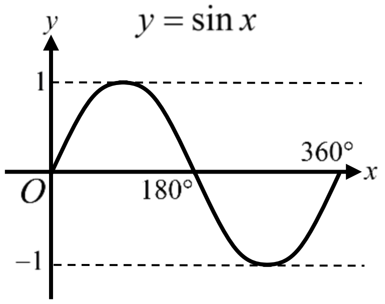
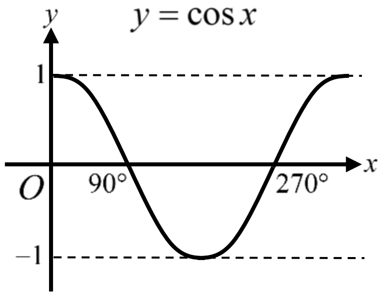
$$\begin{cases} \sin\theta=\dfrac{\mathrm{AB}}{\mathrm{AC}} \\ \cos\theta=\dfrac{\mathrm{BC}}{\mathrm{AC}} \\ \tan\theta=\dfrac{\mathrm{AB}}{\mathrm{BC}} \end{cases} : \text{Trigonometric ratios}$$



✓ Properties of a general trigonometric function $y=A\sin B(x-C)+D$:

1. $A=\dfrac{y_{\max}-y_{\min}}{2}$: Amplitude
2. $B=\dfrac{360^\circ}{\text{Period}}$ or $\dfrac{2\pi}{\text{Period}}$
3. $D=\dfrac{y_{\max}+y_{\min}}{2}$
4. C can be found by substitution of a point on the graph

✓ Properties of graphs of trigonometric functions:

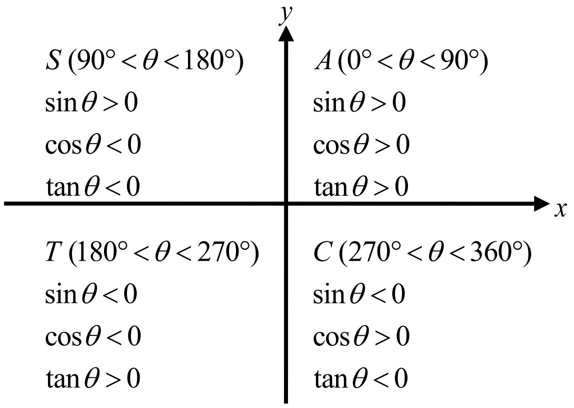
	1. Amplitude $=1$ 2. Period $=360^\circ$ or 2π 3. $-1\le\sin x\le 1$
	1. Amplitude $=1$ 2. Period $=360^\circ$ or 2π 3. $-1\le\cos x\le 1$

✓ Trigonometric identities:

1. $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$

2. $\sin^2\theta + \cos^2\theta \equiv 1$

✓ ASTC diagram



13

2-D Trigonometry

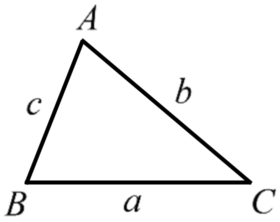
✓ Consider a triangle ABC:

1. $\dfrac{\sin A}{a} = \dfrac{\sin B}{b}$ or $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$: Sine rule

2. $a^2 = b^2 + c^2 - 2bc\cos A$

   or $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$: Cosine rule

3. $\dfrac{1}{2}ab\sin C$: Area of the triangle ABC

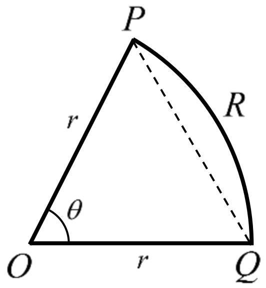


✓ Consider a sector $OPRQ$ with centre O, radius r and $\angle POQ = \theta^\circ$:

$2\pi r \times \dfrac{\theta^\circ}{360^\circ}$: Arc length PRQ

$\pi r^2 \times \dfrac{\theta^\circ}{360^\circ}$: Area of the sector $OPRQ$

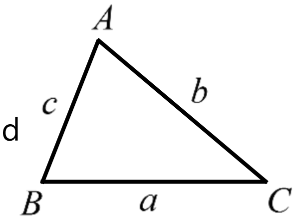
$\pi r^2 \times \dfrac{\theta^\circ}{360^\circ} - \dfrac{1}{2}r^2 \sin\theta^\circ$: Area of the segment PRQ



✓ Consider a triangle ABC:

$\frac{\sin A}{a}=\frac{\sin B}{b}$ or $\frac{a}{\sin A}=\frac{b}{\sin B}$: Sine rule

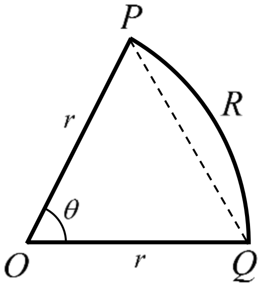
Note: The ambiguous case exists if two sides and an angle are known, and the angle is opposite to the shorter known side



✓ $\frac{x^\circ}{180^\circ}=\frac{y\text{ rad}}{\pi\text{ rad}}$: Method of conversions between degree and radian

✓ Consider a sector $OPRQ$ with centre O, radius r and $\angle POQ=\theta$ in radian:

1. $r\theta$: Arc length PQ
2. $\frac{1}{2}r^2\theta$: Area of the sector $OPRQ$
3. $\frac{1}{2}r^2(\theta-\sin\theta)$: Area of the segment PRQ



14 Areas and Volumes

✓ For a cube of side length l:

1. $6l^2$: Total surface area
2. l^3: Volume

✓ For a cuboid of side lengths a, b and c:

1. $2(ab+bc+ac)$: Total surface area
2. abc: Volume

✓ For a prism of height h and cross-sectional area A:

1. Ah: Volume

✓ For a cylinder of height h and radius r:

1. $2\pi r^2+2\pi rh$: Total surface area
2. $2\pi rh$: Lateral surface area
3. πr^2h: Volume

✓ For a pyramid of height h and base area A:

1. $\frac{1}{3}Ah$: Volume

✓ For a circular cone of height h and radius r:

1. $l=\sqrt{r^2+h^2}$: Slant height
2. $\pi r^2+\pi rl$: Total surface area
3. πrl: Curved surface area
4. $\frac{1}{3}\pi r^2 h$: Volume

✓ For a sphere of radius r:

1. $4\pi r^2$: Total surface area
2. $\frac{4}{3}\pi r^3$: Volume

✓ For a hemisphere of radius r:

1. $3\pi r^2$: Total surface area
2. $2\pi r^2$: Curved surface area
3. $\frac{2}{3}\pi r^3$: Volume

15 Complex Numbers

✓ Terminologies of complex numbers:

$\mathrm{i}=\sqrt{-1}$: Imaginary unit

$z=a+b\mathrm{i}$: Complex number in Cartesian form

a: Real part of z

b: Imaginary part of z

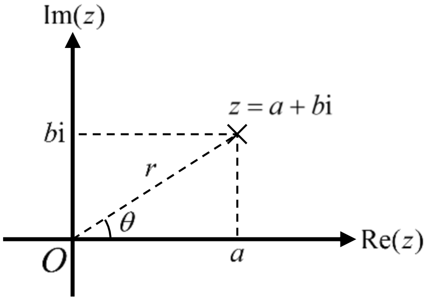
$z^*=a-b\mathrm{i}$: Conjugate of $z=a+b\mathrm{i}$

$|z|=r=\sqrt{a^2+b^2}$: Modulus of $z=a+b\mathrm{i}$

$\arg(z)=\theta=\arctan\frac{b}{a}$: Argument of $z=a+b\mathrm{i}$

✓ Properties of Argand diagram:

1. Real axis: Horizontal axis
2. Imaginary axis: Vertical axis



✓ Forms of complex numbers:

1. $z=a+b\text{i}$: Cartesian form
2. $z=r(\cos\theta+\text{i}\sin\theta)=r\text{cis}\theta$: Modulus-argument form
3. $z=re^{\text{i}\theta}$: Euler form

✓ Properties of moduli and arguments of complex numbers z_1 and z_2:

1. $|z_1z_2|=|z_1||z_2|$
2. $\left|\frac{z_1}{z_2}\right|=\frac{|z_1|}{|z_2|}$
3. $\arg(z_1z_2)=\arg z_1+\arg z_2$
4. $\arg\left(\frac{z_1}{z_2}\right)=\arg z_1-\arg z_2$

✓ If $z=a+b\text{i}$ is a root of the polynomial equation $p(z)=0$, then $z^*=a-b\text{i}$ is also a root of $p(z)=0$

16 Matrices

- ✓ Terminologies of matrices:

$$\mathbf{A}=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$: A $m \times n$ matrix with m rows and n columns

a_{ij} : Element on the i th row and the j th column

$$\mathbf{I}=\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$: Identity matrix

$$\mathbf{0}=\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$: Zero matrix

$$\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}$$: Diagonal matrix

$|\mathbf{A}| = \det(\mathbf{A})$: Determinant of $\mathbf{A}$

$\mathbf{A}$ is non-singular if $\det(\mathbf{A}) \neq 0$

$\mathbf{A}^{-1}$: Inverse of $\mathbf{A}$

$\mathbf{A}^{-1}$ exists if $\mathbf{A}$ is non-singular

- ✓ For any 2×2 square matrices $\mathbf{A}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$:

1. $|\mathbf{A}| = \det(\mathbf{A}) = ad - bc$: Determinant of $\mathbf{A}$
2. $\mathbf{A}^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$: Inverse of $\mathbf{A}$

✓ Operations of matrices:

1. $$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \pm \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix} = \begin{pmatrix} a_{11} \pm b_{11} & \cdots & a_{1n} \pm b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} \pm b_{m1} & \cdots & a_{mn} \pm b_{mn} \end{pmatrix}$$

2. $$k\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} ka_{11} & \cdots & ka_{1n} \\ \vdots & \ddots & \vdots \\ ka_{m1} & \cdots & ka_{mn} \end{pmatrix}$$

3. $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$: The element on the i th row and the j th column of $\mathbf{C} = \mathbf{AB} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}\begin{pmatrix} b_{11} & \cdots & b_{1k} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nk} \end{pmatrix}$, where $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are $m \times n$, $n \times k$ and $m \times k$ matrices respectively

✓ A 2×2 system $\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$ can be expressed as $\mathbf{AX} = \mathbf{B}$, where $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$ can be solved by $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B} = \begin{pmatrix} a & b \\ d & e \end{pmatrix}^{-1}\begin{pmatrix} c \\ f \end{pmatrix}$

✓ A 3×3 system $\begin{cases} ax + by + cz = d \\ ex + fy + gz = h \\ ix + jy + kz = l \end{cases}$ can be expressed as $\mathbf{AX} = \mathbf{B}$, where $\mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ can be solved by $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B} = \begin{pmatrix} a & b & c \\ e & f & g \\ i & j & k \end{pmatrix}^{-1}\begin{pmatrix} d \\ h \\ l \end{pmatrix}$

✓ Eigenvalues and eigenvectors of $\mathbf{A}$:

1. $\det(\mathbf{A} - \lambda\mathbf{I})$: Characteristic polynomial of $\mathbf{A}$
2. Solution(s) of $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$: Eigenvalue(s) of $\mathbf{A}$
3. $\mathbf{v}$: Eigenvector of $\mathbf{A}$ corresponding to the eigenvalue λ, which satisfies $\mathbf{Av} = \lambda\mathbf{v}$

✓ Diagonalization of $\mathbf{A}$:

1. $\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$: Diagonal matrix of the eigenvalues of $\mathbf{A}$
2. $\mathbf{V} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix}$: A matrix of the eigenvectors of $\mathbf{A}$
3. $\mathbf{A} = \mathbf{VDV}^{-1} \Rightarrow \mathbf{A}^n = \mathbf{VD}^n\mathbf{V}^{-1}$

✓ Two-dimensional transformation matrices:

1. $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$: Reflection about the x-axis
2. $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$: Reflection about the y-axis
3. $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$: Reflection about the line $y = mx$, where $m = \tan\theta$
4. $\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$: Vertical stretch with scale factor k
5. $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$: Horizontal stretch with scale factor k
6. $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$: Enlargement about the origin with scale factor k
7. $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$: Rotation with positive angle θ anticlockwise about the origin
8. $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$: Rotation with positive angle θ clockwise about the origin
9. Area of the image $= |\det(T)| \cdot$ Area of the object, where T is the transformation matrix

17 Vectors

✓ Terminologies of vectors:

$\overrightarrow{\mathrm{AB}}$: Vector of length AB with initial point A and terminal point B

$\overrightarrow{\mathrm{OP}}$: Position vector of P, where O is the origin

$\left|\overrightarrow{\mathrm{AB}}\right|$: Magnitude (length) of $\overrightarrow{\mathrm{AB}}$

$\hat{\mathbf{v}} = \frac{1}{|\mathbf{v}|}\mathbf{v}$: Unit vector parallel to $\mathbf{v}$, with $|\hat{\mathbf{v}}| = 1$

$\mathbf{0}$: Zero vector

$\mathbf{i}$: Unit vector along the positive x-axis

$\mathbf{j}$: Unit vector along the positive y-axis

$\mathbf{k}$: Unit vector along the positive z-axis

✓ A vector $\mathbf{v}$ can be expressed as $\mathbf{v}=v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k}$ or $\mathbf{v}=\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

✓ Properties of vectors:

1. $\overrightarrow{\mathrm{AB}}=\overrightarrow{\mathrm{OB}}-\overrightarrow{\mathrm{OA}}$
2. $\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}\pm\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}=\begin{pmatrix} u_1\pm v_1 \\ u_2\pm v_2 \\ u_3\pm v_3 \end{pmatrix}$
3. $\mathbf{v}$ and $k\mathbf{v}$ are in the same direction if $k>0$
4. $\mathbf{v}$ and $k\mathbf{v}$ are in opposite direction if $k<0$
5. $k\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}=\begin{pmatrix} kv_1 \\ kv_2 \\ kv_3 \end{pmatrix}$

✓ Properties of the scalar product $\mathbf{u}\cdot\mathbf{v}$ of $\mathbf{u}=\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ and $\mathbf{v}=\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$:

1. $\mathbf{u}\cdot\mathbf{v}=u_1v_1+u_2v_2+u_3v_3=|\mathbf{u}||\mathbf{v}|\cos\theta$
2. $\mathbf{i}\cdot\mathbf{i}=\mathbf{j}\cdot\mathbf{j}=\mathbf{k}\cdot\mathbf{k}=1$
3. $\mathbf{i}\cdot\mathbf{j}=\mathbf{j}\cdot\mathbf{k}=\mathbf{k}\cdot\mathbf{i}=0$
4. $\mathbf{u}$ and $\mathbf{v}$ are in the same direction if $\mathbf{u}\cdot\mathbf{v}=|\mathbf{u}||\mathbf{v}|$
5. $\mathbf{u}$ and $\mathbf{v}$ are in opposite direction if $\mathbf{u}\cdot\mathbf{v}=-|\mathbf{u}||\mathbf{v}|$
6. $\mathbf{u}$ and $\mathbf{v}$ are perpendicular if $\mathbf{u}\cdot\mathbf{v}=0$
7. $\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u}$
8. $\mathbf{u}\cdot\mathbf{u}=|\mathbf{u}|^2$

- ✓ Properties of the vector product $\mathbf{u}\times\mathbf{v}$ of $\mathbf{u}=\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}$ and $\mathbf{v}=\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}$ where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$:

1. $\mathbf{u}\times\mathbf{v}=\begin{pmatrix}u_2v_3-u_3v_2\\u_3v_1-u_1v_3\\u_1v_2-u_2v_1\end{pmatrix}=|\mathbf{u}||\mathbf{v}|\sin\theta\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}//(\mathbf{u}\times\mathbf{v})$
2. $\mathbf{i}\times\mathbf{i}=\mathbf{j}\times\mathbf{j}=\mathbf{k}\times\mathbf{k}=\mathbf{0}$
3. $\mathbf{i}\times\mathbf{j}=\mathbf{k}$, $\mathbf{j}\times\mathbf{k}=\mathbf{i}$ and $\mathbf{k}\times\mathbf{i}=\mathbf{j}$
4. $\mathbf{j}\times\mathbf{i}=-\mathbf{k}$, $\mathbf{k}\times\mathbf{j}=-\mathbf{i}$ and $\mathbf{i}\times\mathbf{k}=-\mathbf{j}$
5. $\mathbf{u}$ and $\mathbf{v}$ are parallel if $\mathbf{u}\times\mathbf{v}=0$
6. $\mathbf{u}$ and $\mathbf{v}$ are perpendicular if $|\mathbf{u}\times\mathbf{v}|=|\mathbf{u}||\mathbf{v}|$
7. $\mathbf{u}\times\mathbf{v}=-(\mathbf{v}\times\mathbf{u})$

- ✓ The area of the parallelogram with adjacent sides $\overrightarrow{\mathrm{AB}}$ and $\overrightarrow{\mathrm{AD}}$ is $\left|\overrightarrow{\mathrm{AB}}\times\overrightarrow{\mathrm{AD}}\right|$

- ✓ The area of the triangle with adjacent sides $\overrightarrow{\mathrm{AB}}$ and $\overrightarrow{\mathrm{AD}}$ is $\frac{1}{2}\left|\overrightarrow{\mathrm{AB}}\times\overrightarrow{\mathrm{AD}}\right|$

- ✓ Forms of the straight line with fixed point $\mathrm{A}(a_1,a_2,a_3)$ and direction vector $\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$:

1. $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}+t\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$, $t\in\mathbb{R}$
2. $\begin{cases}x=a_1+b_1t\\y=a_2+b_2t\\z=a_3+b_3t\end{cases}$: Parametric form

- ✓ Vector components:

1. $\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}$: Vector component of $\mathbf{u}$ parallel to $\mathbf{v}$
2. $\frac{|\mathbf{u}\times\mathbf{v}|}{|\mathbf{v}|}$: Vector component of $\mathbf{u}$ perpendicular to $\mathbf{v}$

18 Graph Theory

✓ Terminologies of graphs:

Vertex: A point on a graph

Edge: Arcs that connect vertices

Walk: A sequence of edges

Path: A sequence of edges that passes through any vertex and any edge at most once

Degree of a vertex: Number of edges connecting the vertex

Connected graph: A graph that there exists at least one walk between any two vertices

Unconnected graph: A graph that there exist at least two vertices that there is no walk between them

Subgraph of a graph: A collection of some edges and vertices of the original graph

Loop: An edge that starts and ends at the same vertex

Simple graph: A graph that has no loops and no multiple edges connecting the same pair of vertices

Multiple graph: A graph that has multiple edges connecting at least one pair of vertices

Cycle: A path that the starting vertex is the end vertex

Tree: A connected graph with no cycles

Spanning tree: A tree that connects all vertices in the graph

✓ Directed graphs:

1. Directed graph: A graph that all edges are assigned with directions
2. In-degree of a vertex: Number of edges connecting and pointing towards the vertex
3. Out-degree of a vertex: Number of edges connecting and pointing away from the vertex

- ✓ Adjacency matrix $\mathbf{M}$ of a graph with n vertices:
    1. $n \times n$: Order of $\mathbf{M}$
    2. The entry $m_{ij} = 1$ if there is an edge connecting the vertex i and the vertex j, and $m_{ij} = 0$ if otherwise
    3. $\mathbf{M}^p$ shows the number of walks of length p in the graph
    4. $\sum_{r=1}^{p} \mathbf{M}^r$ shows the number of walks of length less than or equal to p in the graph
    5. The column sum of a transition matrix of a directed graph must be equal to 1

- ✓ Algorithms of finding minimum spanning trees:
    1. Kruskal's algorithm
    2. Prim's algorithm

- ✓ Eulerian trails and circuits:
    1. Trail: A sequence of edges that passes through any edge at most once
    2. Circuit: A trail that the starting vertex is the end vertex
    3. Eulerian trail: A trail that passes through all edges of a graph
    4. Eulerian circuit: A circuit that passes through all edges of a graph
    5. An Eulerian trail exists if there exists two and only two vertices of odd degree
    6. An Eulerian circuit exists if all vertices are of even degree
    7. Chinese postman problem can be used to find the route of minimum weight that covers all edges of a graph

- ✓ Hamiltonian paths and cycles:
    1. Complete graph: A graph that there exists an edge for any pair of two vertices
    2. Hamiltonian path: A path that passes through all vertices of a graph
    3. Hamiltonian cycle: A cycle that passes through all vertices of a graph

- ✓ Travelling Salesman problem:
    1. Travelling Salesman problem can be used to find the cycle of minimum weight that passes through all vertices of a graph
    2. Nearest neighbour algorithm can be used to find the upper bound of the solution of a travelling salesman problem
    3. Deleted vertex algorithm can be used to find the lower bound of the solution of a travelling salesman problem

19 Differentiation

✓ $\frac{\mathrm{d}y}{\mathrm{d}x}=f'(x)$: Derivative of the function $y=f(x)$ (First derivative)

✓ Rules of differentiation:

1. $f(x)=x^n \Rightarrow f'(x)=nx^{n-1}$
2. $f(x)=p(x)+q(x) \Rightarrow f'(x)=p'(x)+q'(x)$
3. $f(x)=cp(x) \Rightarrow f'(x)=cp'(x)$

✓ Relationship between graph properties and the derivatives:

1. $f'(x)>0$ for $a\le x\le b$: $f(x)$ is increasing in the interval
2. $f'(x)<0$ for $a\le x\le b$: $f(x)$ is decreasing in the interval
3. $f'(a)=0$: $(a, f(a))$ is a stationary point of $f(x)$
4. $f'(a)=0$ and $f'(x)$ changes from positive to negative at $x=a$: $(a, f(a))$ is a maximum point of $f(x)$
5. $f'(a)=0$ and $f'(x)$ changes from negative to positive at $x=a$: $(a, f(a))$ is a minimum point of $f(x)$

✓ Tangents and normals:

1. $f'(a)$: Slope of tangent at $x=a$
2. $\frac{-1}{f'(a)}$: Slope of normal at $x=a$
3. $y-f(a)=f'(a)(x-a)$: Equation of tangent at $x=a$
4. $y-f(a)=\left(\frac{-1}{f'(a)}\right)(x-a)$: Equation of normal at $x=a$

✓ Derivatives of a function $y=f(x)$:

1. $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=f''(x)$: Second derivative
2. $\frac{\mathrm{d}^ny}{\mathrm{d}x^n}=f^{(n)}(x)$: n-th derivative

✓ More rules of differentiation:

$f(x)=\sin x \Rightarrow f'(x)=\cos x$	$f(x)=p(q(x)) \Rightarrow f'(x)=p'(q(x))\cdot q'(x)$
$f(x)=\cos x \Rightarrow f'(x)=-\sin x$	$f(x)=p(x)q(x)$ $\Rightarrow f'(x)=p'(x)q(x)+p(x)q'(x)$
$f(x)=\tan x \Rightarrow f'(x)=\dfrac{1}{\cos^2 x}$	
$f(x)=e^x \Rightarrow f'(x)=e^x$	$f(x)=\dfrac{p(x)}{q(x)}$ $\Rightarrow f'(x)=\dfrac{p'(x)q(x)-p(x)q'(x)}{(q(x))^2}$
$f(x)=\ln x \Rightarrow f'(x)=\dfrac{1}{x}$	

✓ $f''(a)=0$ and $f''(x)$ changes sign at $x=a$: $(a, f(a))$ is a point of inflexion of $f(x)$

✓ $\dfrac{\mathrm{d}N}{\mathrm{d}t}=\dfrac{\mathrm{d}N}{\mathrm{d}x}\cdot\dfrac{\mathrm{d}x}{\mathrm{d}t}$: Rate of change of N with respect to the time t

✓ Tests for optimization:

1. First derivative test
2. Second derivative test

✓ Applications in kinematics:

1. $s(t)$: Displacement with respect to the time t
2. $v(t)=s'(t)$: Velocity
3. $a(t)=v'(t)$: Acceleration

20 Integration and Trapezoidal Rule

✓ Integrals of a function $y=f(x)$:

1. $\int f(x)\mathrm{d}x$: Indefinite integral of $f(x)$
2. $\int_a^b f(x)\mathrm{d}x$: Definite integral of $f(x)$ from a to b

✓ Rules of integration:

1. $\int x^n \mathrm{d}x = \frac{1}{n+1}x^{n+1} + C$
2. $\int (p'(x) + q'(x))\mathrm{d}x = p(x) + q(x) + C$
3. $\int cp'(x)\mathrm{d}x = cp(x) + C$

✓ $\int_a^b f(x)\mathrm{d}x$: Area under the graph of $f(x)$ and above the x-axis, between $x = a$ and $x = b$

✓ Trapezoidal Rule:

a, b ($a < b$): End points

n: Number of intervals

$h = \frac{b-a}{n}$: Interval width

$\int_a^b f(x)\mathrm{d}x$ can be estimated by $\frac{1}{2}h\left[f(x_0) + f(x_n) + 2(f(x_1) + f(x_2) + \ldots + f(x_{n-1}))\right]$

✓ Estimation by Trapezoidal Rule:

1. The estimation overestimates if the estimated value is greater than the actual value of $\int_a^b f(x)\mathrm{d}x$
2. The estimation underestimates if the estimated value is less than the actual value of $\int_a^b f(x)\mathrm{d}x$

✓ More rules of integration:

$\int \sin x\mathrm{d}x = -\cos x + C$	$\int e^x \mathrm{d}x = e^x + C$
$\int \cos x\mathrm{d}x = \sin x + C$	$\int \frac{1}{x}\mathrm{d}x = \ln x + C$
$\int \frac{1}{\cos^2 x}\mathrm{d}x = \tan x + C$	Integration by substitution
$\int_a^b f'(x)\mathrm{d}x = \left[f(x)\right]_a^b = f(b) - f(a)$	

✓ Areas on x-y plane, between $x = a$ and $x = b$:

1. $\int_a^b \lvert f(x)\rvert \mathrm{d}x$: Area between the graph of $f(x)$ and the x-axis
2. $\int_a^b \lvert f(x) - g(x)\rvert \mathrm{d}x$: Area between the graph of $f(x)$ and the graph of $g(x)$

✓ Areas on x-y plane, between $y=c$ and $y=d$:

1. $\int_c^d |g(y)|\mathrm{d}y$: Area between the graph of $g(y)$ and the y-axis
2. $\int_c^d |g(y)-f(y)|\mathrm{d}y$: Area between the graph of $g(y)$ and the graph of $f(y)$

✓ Volumes of revolutions about the x-axis, between $x=a$ and $x=b$:

1. $V=\pi\int_a^b (f(x))^2\mathrm{d}x$: Volume of revolution when the region between the graph of $f(x)$ and the x-axis is rotated 360° about the x-axis
2. $V=\pi\int_a^b ((f(x))^2-(g(x))^2)\mathrm{d}x$: Volume of revolution when the region between the graphs of $f(x)$ and $g(x)$ is rotated 360° about the x-axis

✓ Volumes of revolutions about the y-axis, between $y=c$ and $y=d$:

1. $V=\pi\int_c^d (g(y))^2\mathrm{d}y$: Volume of revolution when the region between the graph of $g(y)$ and the y-axis is rotated 360° about the y-axis
2. $V=\pi\int_c^d ((g(y))^2-(f(y))^2)\mathrm{d}y$: Volume of revolution when the region between the graphs of $g(y)$ and $f(y)$ is rotated 360° about the y-axis

✓ Applications in kinematics:

1. $a(t)$: Acceleration with respect to the time t
2. $v(t)=\int a(t)\mathrm{d}t$: Velocity
3. $s(t)=\int v(t)\mathrm{d}t$: Displacement
4. $d=\int_{t_1}^{t_2}|v(t)|\mathrm{d}t$: Total distance travelled between t_1 and t_2

21 Differential Equations

✓ $\frac{\mathrm{d}y}{\mathrm{d}x}=f(x,y)$: First order differential equation

✓ Solving $\frac{\mathrm{d}y}{\mathrm{d}x}=f(x)g(y)$ by separating variables:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=f(x)g(y)\Rightarrow\int\frac{1}{g(y)}\mathrm{d}y=\int f(x)\mathrm{d}x$$

- ✓ Coupled differential equations:

1. $\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = ax + by \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = cx + dy \end{cases}$ can be expressed as $\begin{pmatrix} \dfrac{\mathrm{d}x}{\mathrm{d}t} \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$
2. λ_1, λ_2: Eigenvalues of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$
3. $\mathbf{v}_1$, $\mathbf{v}_2$: Eigenvectors corresponding to λ_1 and λ_2 respectively
4. $\mathbf{x} = Ae^{\lambda_1 t}\mathbf{v}_1 + Be^{\lambda_2 t}\mathbf{v}_2$: Solution of the system
5. Stable equilibrium if λ_1, $\lambda_2 < 0$ or $\lambda_1 = a + b\mathrm{i}$, $\lambda_2 = a - b\mathrm{i}$ and $a < 0$
6. Unstable equilibrium if λ_1, $\lambda_2 > 0$ or $\lambda_1 = a + b\mathrm{i}$, $\lambda_2 = a - b\mathrm{i}$ and $a > 0$
7. Saddle point if $\lambda_1 \lambda_2 < 0$

- ✓ Solving $\dfrac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$ by Euler's method, with (x_0, y_0) and step length h:

$$\begin{cases} x_{n+1} = x_n + h \\ y_{n+1} = y_n + h \left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_{(x_n,\, y_n)} \end{cases}$$

- ✓ Solving $\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = ax + by \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = cx + dy \end{cases}$ by Euler's method, with (t_0, x_0, y_0) and step length h:

$t_{n+1} = t_n + h$ and $\begin{cases} x_{n+1} = x_n + h \left.\dfrac{\mathrm{d}x}{\mathrm{d}t}\right|_{(t_n,\, x_n,\, y_n)} \\ y_{n+1} = y_n + h \left.\dfrac{\mathrm{d}y}{\mathrm{d}t}\right|_{(t_n,\, x_n,\, y_n)} \end{cases}$

- ✓ Predator-prey models:

$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = (a - by)x \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = (cx - d)y \end{cases}$, where a, b, c and d are positive constants

- ✓ The second-order differential equation $\frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx = 0$ can be expressed as

$$\begin{cases} \frac{dv}{dt} = -av - bx \\ \frac{dx}{dt} = v \end{cases}$$

22 Statistics

- ✓ Relationship between frequencies and cumulative frequencies:

Data	Frequency	Data less than or equal to	Cumulative frequency
10	f_1	10	f_1
20	f_2	20	$f_1 + f_2$
30	f_3	30	$f_1 + f_2 + f_3$

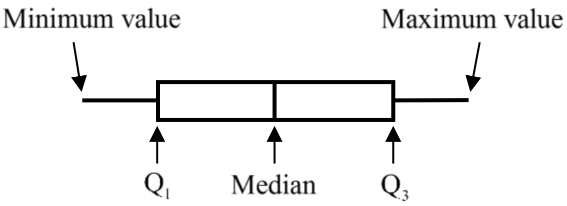
- ✓ Measures of central tendency for a data set $\{x_1, x_2, x_3, \cdots, x_n\}$ arranged in ascending order:

1. $\overline{x} = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}$: Mean
2. The datum or the average value of two data at the middle: Median
3. The datum appears the most: Mode

- ✓ Measures of dispersion for a data set $\{x_1, x_2, x_3, \cdots, x_n\}$ arranged in ascending order:

1. $x_n - x_1$: Range
2. Two subgroups A and B can be formed from the data set such that all data of the subgroup A are less than or equal to the median, while all data of the subgroup B are greater than or equal to the median
3. $Q_1 =$ The median of the subgroup A: Lower quartile
4. $Q_3 =$ The median of the subgroup B: Upper quartile
5. $Q_3 - Q_1$: Inter-quartile range (IQR)
6. $\sigma = \sqrt{\frac{(x_1 - \overline{x})^2 + (x_2 - \overline{x})^2 + (x_3 - \overline{x})^2 + \cdots + (x_n - \overline{x})^2}{n}}$: Standard deviation

✓ Box-and-whisker diagram:



✓ A datum x is defined to be an outlier if $x < Q_1 - 1.5\text{IQR}$ or $x > Q_3 + 1.5\text{IQR}$

✓ Coding of data:

1. Only the mean, the median, the mode and the quartiles will change when each datum of the data set is added or subtracted by a value
2. All measures of central tendency and measures of dispersion will change when each datum of the data set is multiplied or divided by a value

23 Probability

✓ Terminologies:

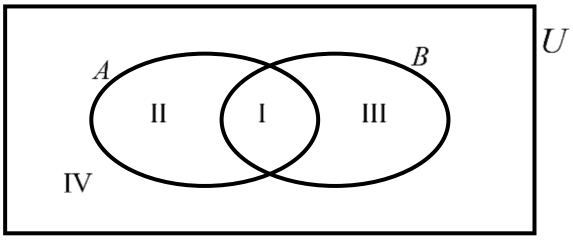
1. U : Universal set
2. A : Event
3. x : Outcome of an event
4. $n(U)$: Total number of elements
5. $n(A)$: Number of elements in A

✓ Formulae for probability:

1. $\text{P}(A \cup B) = \text{P}(A) + \text{P}(B) - \text{P}(A \cap B)$
2. $\text{P}(A') = 1 - \text{P}(A)$
3. $\text{P}(A \mid B) = \dfrac{\text{P}(A \cap B)}{\text{P}(B)}$
4. $\text{P}(A) = \text{P}(A \cap B) + \text{P}(A \cap B')$
5. $\text{P}(A' \cap B') + \text{P}(A \cup B) = 1$
6. $\text{P}(A \cup B) = \text{P}(A) + \text{P}(B)$ and $\text{P}(A \cap B) = 0$ if A and B are mutually exclusive
7. $\text{P}(A \cap B) = \text{P}(A) \cdot \text{P}(B)$ and $\text{P}(A \mid B) = \text{P}(A)$ if A and B are independent

✓ Venn diagram:

1. Region I: $A \cap B$
2. Region II: $A \cap B'$
3. Region III: $A' \cap B$
4. Region IV: $(A \cup B)'$



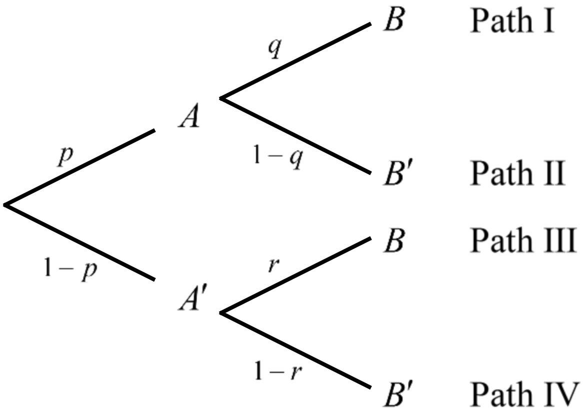
✓ Tree diagram:

1. Path I: $\mathrm{P}(A \cap B) = pq$
2. Path I + Path III:

$= \mathrm{P}(B)$

$= \mathrm{P}(A \cap B) + \mathrm{P}(A' \cap B)$

$= pq + (1-p)r$



✓ Markov Chain with transition matrix $\mathbf{T}$:

1. $\det(\mathbf{T} - \lambda \mathbf{I})$: Characteristic polynomial of $\mathbf{T}$
2. Solution(s) of $\det(\mathbf{T} - \lambda \mathbf{I}) = 0$: Eigenvalue(s) of $\mathbf{T}$
3. $\mathbf{v}$: Steady state probability vector, which is the eigenvector of $\mathbf{T}$ corresponding to the eigenvalue $\lambda = 1$
4. $\mathbf{v}_0$: Initial state probability vector
5. $\mathbf{v}_n = \mathbf{T}^n \mathbf{v}_0$: State probability vector after n transitions
6. The column sum of $\mathbf{T}$ must be equal to 1

24 Discrete Probability Distributions

✓ Properties of a discrete random variable X:

X	x_1	x_2	...	x_n
$\mathrm{P}(X = x)$	$\mathrm{P}(X = x_1)$	$\mathrm{P}(X = x_2)$	...	$\mathrm{P}(X = x_n)$

1. $\mathrm{P}(X = x_1) + \mathrm{P}(X = x_2) + \cdots + \mathrm{P}(X = x_n) = 1$
2. $\mathrm{E}(X) = x_1 \mathrm{P}(X = x_1) + x_2 \mathrm{P}(X = x_2) + \cdots + x_n \mathrm{P}(X = x_n)$: Expected value of X
3. $\mathrm{E}(X) = 0$ if a fair game is considered

25 Binomial Distribution

✓ Properties of a random variable $X \sim \mathrm{B}(n, p)$ following binomial distribution:

1. Only two outcomes from every independent trial (Success and failure)
2. n: Number of trials
3. p: Probability of success
4. X: Number of successes in n trials

✓ Formulae for binomial distribution:

1. $\mathrm{P}(X = r) = \binom{n}{r} p^r (1-p)^{n-r}$ for $0 \le r \le n$, $r \in \overline{\mathbb{Z}}$
2. $\mathrm{E}(X) = np$: Expected value of X
3. $\mathrm{Var}(X) = np(1-p)$: Variance of X
4. $\sqrt{np(1-p)}$: Standard deviation of X
5. $\mathrm{P}(X \le r) = \mathrm{P}(X < r+1) = 1 - \mathrm{P}(X \ge r+1)$

26 Poisson Distribution

✓ Properties of a random variable $X \sim \mathrm{Po}(\lambda)$ following Poisson distribution:

1. The expected number of occurrences of an event is directly proportional to the length of the time interval
2. The numbers of occurrences of the event in different disjoint time intervals are independent
3. λ: Mean number of occurrences of an event
4. X: Number of occurrences of an event

✓ Formulae for Poisson distribution:

1. $\mathrm{P}(X = r) = \dfrac{e^{-\lambda} \cdot \lambda^r}{r!}$ for $r \ge 0$, $r \in \mathbb{Z}$
2. $\mathrm{E}(X) = \lambda$: Expected value of X
3. $\mathrm{Var}(X) = \lambda$: Variance of X
4. $\sqrt{\lambda}$: Standard deviation of X
5. $\mathrm{P}(X \le r) = \mathrm{P}(X < r+1) = 1 - \mathrm{P}(X \ge r+1)$

27 Normal Distribution

✓ Properties of a random variable $X \sim \mathrm{N}(\mu, \sigma^2)$ following normal distribution:

1. μ: Mean
2. σ: Standard deviation
3. The mean, the median and the mode are the same
4. The normal curve representing the distribution is a bell-shaped curve which is symmetric about the middle vertical line
5. $\mathrm{P}(X < \mu) = \mathrm{P}(X > \mu) = 0.5$
6. The total area under the curve is 1

28 Linear Combinations of Variables

✓ Properties of linear transformations of independent random variables:

1. $\mathrm{E}(aX+b) = a\mathrm{E}(X)+b$
2. $\mathrm{Var}(aX+b) = a^2\mathrm{Var}(X)$
3. $\mathrm{E}(a_1X_1 + a_2X_2 + \cdots + a_nX_n) = a_1\mathrm{E}(X_1) + a_2\mathrm{E}(X_2) + \cdots + a_n\mathrm{E}(X_n)$
4. $\mathrm{Var}(a_1X_1 + a_2X_2 + \cdots + a_nX_n) = a_1^2\mathrm{Var}(X_1) + a_2^2\mathrm{Var}(X_2) + \cdots + a_n^2\mathrm{Var}(X_n)$

29 Point Estimation

✓ Central limit theorem:

1. $X_i \sim N(\mu, \sigma^2)$
2. $\bar{X} = \dfrac{X_1 + X_2 + \cdots + X_n}{n}$: Sample mean
3. $\bar{X} \sim N\left(\mu, \dfrac{\sigma^2}{n}\right)$ when n is sufficiently large

✓ Properties of point estimation:

1. $s_{n-1}^2=\frac{\sum_{i=1}^{n}(X_i-\overline{X})^2}{n-1}$: Sample variance
2. $s_n^2=\frac{\sum_{i=1}^{n}(X_i-\overline{X})^2}{n}$
3. $s_{n-1}^2=\frac{n}{n-1}s_n^2$
4. $\mathrm{E}(\overline{X})=X$
5. $\mathrm{E}(s_{n-1}^2)=\sigma^2$

30 Interval Estimation

✓ $(1-\alpha)\%$ confidence interval for population mean μ when the population variance σ^2 is known:

1. $\overline{X}=\frac{X_1+X_2+\cdots+X_n}{n}$: Sample mean
2. $\left(\overline{X}-z_{\frac{\alpha}{2}}\cdot\frac{\sigma}{\sqrt{n}},\overline{X}+z_{\frac{\alpha}{2}}\cdot\frac{\sigma}{\sqrt{n}}\right)$, where $P\left(Z>z_{\frac{\alpha}{2}}\right)=\frac{\alpha}{2}$
3. $2z_{\frac{\alpha}{2}}\cdot\frac{\sigma}{\sqrt{n}}$: Interval width

✓ $(1-\alpha)\%$ confidence interval for population mean μ when the population variance σ^2 is unknown:

1. $\overline{X}=\frac{X_1+X_2+\cdots+X_n}{n}$: Sample mean
2. $\left(\overline{X}-t_{n-1,\frac{\alpha}{2}}\cdot\frac{s_{n-1}}{\sqrt{n}},\overline{X}+t_{n-1,\frac{\alpha}{2}}\cdot\frac{s_{n-1}}{\sqrt{n}}\right)$, where $P\left(\overline{X}>t_{n-1,\frac{\alpha}{2}}\right)=\frac{\alpha}{2}$
3. $2t_{n-1,\frac{\alpha}{2}}\cdot\frac{s_{n-1}}{\sqrt{n}}$: Interval width

31 Bivariate Analysis

✓ Correlations:

Positive	Strong	$0.75 < r < 1$
	Moderate	$0.5 < r < 0.75$
	Weak	$0 < r < 0.5$
No		$r = 0$
Negative	Weak	$-0.5 < r < 0$
	Moderate	$-0.75 < r < -0.5$
	Strong	$-1 < r < -0.75$

where r is the correlation coefficient

✓ Correlation Coefficient for ranked data:

r_s : Spearman's Rank Correlation Coefficient

✓ Coefficient of determination:

R^2 : Coefficient of determination

R^2 % of the variability of the data can be explained by the regression model

✓ Sum of square residuals:

x	x_1	x_2	$\dots$	x_n
y	y_1	y_2	$\dots$	y_n
Predicted value of y	$\hat{y}_1$	$\hat{y}_2$	$\dots$	$\hat{y}_n$

$SS_{res} = \frac{1}{n}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2$: Sum of square residuals

✓ Non-linear regressions:

1. Quadratic regression
2. Cubic regression
3. Quartic regression
4. Exponential regression
5. Logarithmic regression
6. Power regression
7. Logistic regression

32 Statistical Tests

- ✓ Hypothesis test:
  H_0: Null hypothesis
  H_1: Alternative hypothesis
  C: Critical value in the hypothesis test
  α: Significance level

- ✓ χ^2 test for independence for a contingency table with r rows and c columns:
  $n = rc$: Total number of data
  O_i $(i = 1, 2, \ldots, n)$: Observed frequencies
  E_i $(i = 1, 2, \ldots, n)$: Expected frequencies
  $v = (r-1)(c-1)$: Degree of freedom
  $\chi^2_{calc} = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}$: χ^2 test statistic
  H_0: Two variables are independent
  H_1: Two variables are not independent
  H_0 is rejected if $\chi^2_{calc} > C$ or the p-value is less than the significance level
  H_0 is not rejected if $\chi^2_{calc} < C$ or the p-value is greater than the significance level

- ✓ χ^2 goodness of fit test for a contingency table with 1 row and c columns:
  $v = c - 1$: Degree of freedom
  H_0: The data follows an assigned distribution
  H_1: The data does not follow an assigned distribution
  H_0 is rejected if $\chi^2_{calc} > C$ or the p-value is less than the significance level
  H_0 is not rejected if $\chi^2_{calc} < C$ or the p-value is greater than the significance level

- ✓ Two sample t test:
  μ_1, μ_2: The population means of two groups of data
  H_0: $\mu_1 = \mu_2$
  H_1: $\mu_1 > \mu_2$, $\mu_1 < \mu_2$ (for 1-tailed test), $\mu_1 \neq \mu_2$ (for 2-tailed test)
  H_0 is rejected if the p-value is less than the significance level
  H_0 is not rejected if the p-value is greater than the significance level

- ✓ More about χ^2 test for independence and χ^2 goodness of fit test:
  All expected frequencies E_i should be greater than 5

- ✓ Z test when the population variance σ^2 is known:
  μ: Population mean
  H_0: $\mu = \mu_0$
  H_1: $\mu > \mu_0$, $\mu < \mu_0$ (for 1-tailed test), $\mu \neq \mu_0$ (for 2-tailed test)
  H_0 is rejected if the p-value is less than the significance level
  H_0 is not rejected if the p-value is greater than the significance level

- ✓ One sample t test when the population variance σ^2 is unknown:
  μ: Population mean
  H_0: $\mu = \mu_0$
  H_1: $\mu > \mu_0$, $\mu < \mu_0$ (for 1-tailed test), $\mu \neq \mu_0$ (for 2-tailed test)
  H_0 is rejected if the p-value is less than the significance level
  H_0 is not rejected if the p-value is greater than the significance level

- ✓ Paired t test:
  $d = x - y$: Difference between each pair of data from two variables x and y
  H_0: $\mu_d = 0$
  H_1: $\mu_d > 0$, $\mu_d < 0$ (for 1-tailed test), $\mu_d \neq 0$ (for 2-tailed test)
  H_0 is rejected if the p-value is less than the significance level
  H_0 is not rejected if the p-value is greater than the significance level

- ✓ Test involving binomial distribution:
  $X \sim \mathrm{B}(n, p)$
  H_0: $p = p_0$
  H_1: $p > p_0$, $p < p_0$
  x: Observed value of X
  $\mathrm{P}(X \geq x)$ for H_1: $p > p_0$ or $\mathrm{P}(X \leq x)$ for H_1: $p < p_0$: p-value under $X \sim \mathrm{B}(n, p_0)$
  H_0 is rejected if the p-value is less than the significance level
  H_0 is not rejected if the p-value is greater than the significance level

✓ Test involving Poisson distribution:

$X \sim \mathrm{Po}(\lambda)$

H_0: $\lambda = \lambda_0$

H_1: $\lambda > \lambda_0$, $\lambda < \lambda_0$

x: Observed value of X

$\mathrm{P}(X \geq x)$ for H_1: $\lambda > \lambda_0$ or $\mathrm{P}(X \leq x)$ for H_1: $\lambda < \lambda_0$: p-value under $X \sim \mathrm{Po}(\lambda_0)$

H_0 is rejected if the p-value is less than the significance level

H_0 is not rejected if the p-value is greater than the significance level

✓ Test involving bivariate normal distribution:

ρ: Product moment correlation coefficient

H_0: $\rho = 0$

H_1: $\rho > 0$, $\rho < 0$ (for 1-tailed test), $\rho \neq 0$ (for 2-tailed test)

H_0 is rejected if the p-value is less than the significance level

H_0 is not rejected if the p-value is greater than the significance level

✓ Type I and type II errors:

α: Significance level

$\mathrm{P}(\text{Reject } H_0 \mid H_0 \text{ is true}) = \alpha$: Type I error

$\mathrm{P}(\text{Not reject } H_0 \mid H_0 \text{ is not true})$: Type II error

33 Paper 3 Analysis

✓ Nature of paper: Structured question

✓ Time allowed: 60 minutes

✓ Maximum mark: 55 marks

✓ Number of questions: 2

✓ Mark range per question: 25 marks to 30 marks

✓ Weighting: 20% of the total mark

- ✓ Ways of assessing:
  1. Identify a hypothesis test
  2. State
     - (a) a condition
     - (b) a reason
     - (c) an assumption
  3. Write down
     - (a) the value of a quantity
     - (b) the formula of a quantity
  4. Find
     - (a) the value of a quantity
     - (b) the formula of a quantity
  5. Solve an equation
  6. Perform a hypothesis test
  7. Show
     - (a) a quantity equals to a value
     - (b) the formula of a quantity
  8. Estimate the value of a quantity
  9. Predict the value of a quantity
  10. Sketch a graph
  11. Suggest an improvement
  12. Explain a reason
  13. Describe a result
  14. Verify
      - (a) the value of a quantity
      - (b) the expression of a quantity
  15. Comment on
      - (a) the validity of an argument
      - (b) a statement

Notes

Applications and Interpretation Higher Level for IBDP Mathematics
Practice Paper Set 1 – Paper 1 (120 Minutes)

Question – Answer Book

Instructions

1. Attempt **ALL** questions. Write your answers in the spaces provided in this Question - Answer Book.
2. A graphic display calculator is needed.
3. You are suggested to prepare a formula booklet of Applications and Interpretation for IBDP Mathematics when attempting the questions.
4. Supplementary answer sheets and graph papers will be supplied on request.
5. Unless otherwise specified, **ALL** working must be clearly shown.
6. Unless otherwise specified, numerical answers should be either **EXACT** or correct to **3 SIGNIFICANT FIGURES**.
7. The diagrams in this paper are **NOT** necessarily drawn to scale.
8. Information to be read before you start the exam:

	Marker's Use Only	**Examiner's Use Only**	
Question Number	**Marks**	**Marks**	**Maximum Mark**
1			**5**
2			**5**
3			**6**
4			**5**
5			**6**
6			**6**
7			**7**
8			**6**
9			**6**
10			**5**
11			**8**
12			**7**
13			**6**
14			**7**
15			**7**
16			**5**
17			**7**
18			**6**
Overall			
Paper 1 Total			**110**

1. In a football match, eight players take penalty kicks one by one. The table below shows the ball speed of each penalty kick:

Player	Ball Speed	Player	Ball Speed
Abraham	80 kmh^{-1}	Essien	40 kmh^{-1}
Berg	76 kmh^{-1}	Flores	116 kmh^{-1}
Clyne	100 kmh^{-1}	Gana	90 kmh^{-1}
Denayer	66 kmh^{-1}	Harry	76 kmh^{-1}

(a) Find the mean ball speed.

[2]

(b) Write down

(i) the median speed;

(ii) the standard deviation of the speeds;

(iii) the range of the speeds.

[3]

2. The number of seats in a theatre is investigated. The number of seats in the first row of the theatre u_1 is 100. The number of seats in each subsequent row forms an arithmetic sequence. The number of seats in the tenth row u_{10} is 181.

(a) Find the value of d, the common difference.

[2]

(b) Hence, write down the number of seats in the thirteenth row.

[1]

There are 15 rows in the theatre.

(c) Find the **total** number of seats in the theatre.

[2]

3. A triangular park ABC is built such that $\mathrm{AB} = 28\ \mathrm{m}$, $\mathrm{AC} = 32\ \mathrm{m}$ and $\mathrm{BC} = 41\ \mathrm{m}$.

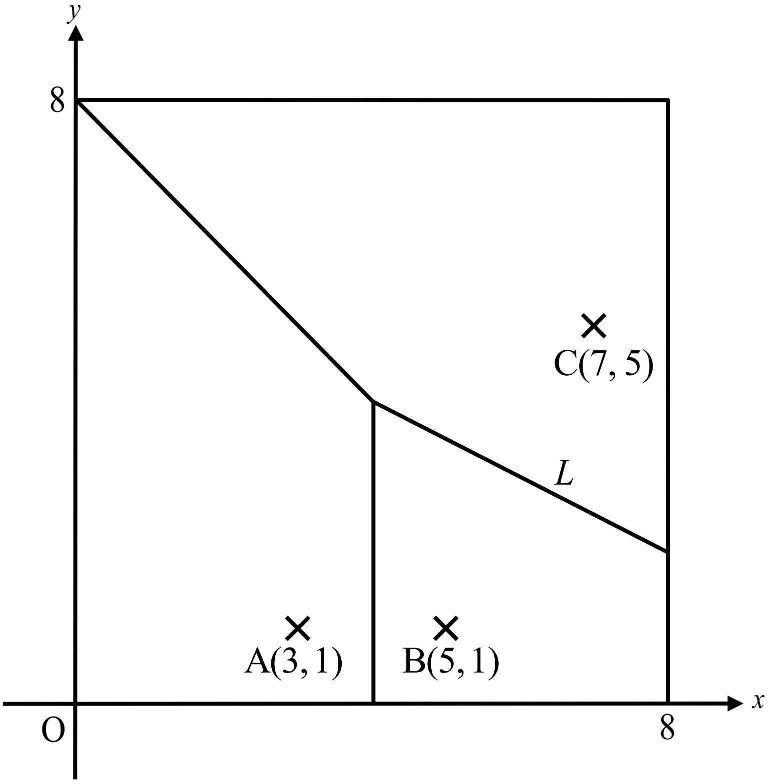
(a) Find $\mathrm{A\hat{B}C}$.

[3]

(b) Hence, find the area of the park.

[3]

4. The diagram below shows the Voronoi diagram of three restaurants for take-away meals, A , B and C , in a town bounded by the coordinate axes, the lines $x = 8$ and $y = 8$, where 1 unit represents 1 km .



The straight line L is the boundary separating the Voronoi cells of B and C . It is given that $(4, 4)$ is a point on L .

(a) (i) Find the gradient of L .

(ii) Hence, find the equation of L , giving the answer in slope-intercept form.

[4]

Kimberly would like to find a restaurant closest to her office to minimize the delivery time of her meal during lunchtime. The position of her office is at $(7, 2.5)$.

(b) State the reason that she is indifferent from choosing the restaurant B and the restaurant C .

[1]

5. A fair eight-faced die with numbered faces $1, 2, 3, 4, 5, 6, 7$ and 8 is tossed for thirteen times.

(a) Find the expected number of multiples of 3 landed.

[2]

(b) Find the variance of the number of multiples of 3 landed.

[2]

(c) Find the probability that the number of multiples of 3 landed is 8.

[2]

6. A closed rectangular box has length $4x$ cm, width $2x$ cm and height y cm, where x, $y > 0$. It is given that the sum of the length and the height of the rectangular box is 20 cm.

(a) Write down

(i) an expression for y in terms of x;

(ii) the possible range of values of x.

[2]

(b) Express V in terms of x, where V cm^3 is the volume of the rectangular box.

[2]

(c) Using the graphic display calculator to find its maximum volume.

[2]

7. Mitsuhide is going to purchase a boat. He is suggested a plan to repay the loan of $\$950000$:

A total of 120 equal monthly payments have to be paid at the end of each month, with a nominal annual interest rate of 3.3%, compounded monthly.

(a) Find the amount of monthly payment. [3]

(b) Find the total amount to be paid. [2]

(c) Hence, find the amount of interest paid. [2]

8. The graph of a quadratic function has y-intercept 150 and one of its x-intercept is -5. The x-coordinate of the vertex of the graph is 5. The equation of the quadratic function is in the form $y = ax^2 + bx + c$.

(a) Write down the value of c.

[1]

(b) Write down the second x-intercept of the function.

[1]

(c) Find the value of a and of b.

[4]

9. In a supermarket, the weights of apples are normally distributed with mean 140 g and standard deviation 9 g, and the weights of oranges are normally distributed with mean 200 g and standard deviation 14 g. Three apples are randomly chosen. Let X be the total weight of the selected apples.

(a) Write down

(i) the mean of X;

(ii) the variance of X.

[2]

Three apples and seven oranges are randomly chosen. Let Y be the total weight of the selected fruits.

(b) Write down

(i) the mean of Y;

(ii) the standard deviation of Y.

[2]

(c) Hence, find $\mathrm{P}(Y \geq 1770)$.

[2]

10. The weight of a plate of area $A\ \mathrm{cm}^2$ is W g. It is given that W varies directly as $\sqrt[3]{A}$. When $A=512$, $W=96$.

(a) Express W in terms of A.

[2]

(b) Write down the area of a plate of weight 60 grams.

[1]

The graph of W is transformed to the new graph of $W=7+24\sqrt[3]{A}$ by two transformations.

(c) Describe geometrically for the two transformations.

[2]

11. The number of torches sold in a store each week follows a Poisson distribution with mean λ, where $\lambda > 20$, $\lambda \in \mathbb{Z}$. The probability that 25 torches are sold in a particular week is 0.0555460.

(a) Find λ.

[2]

(b) Hence, find the probability that

(i) at least 19 torches are sold on a particular week;

(ii) exactly 1 torch is sold on a particular day.

(iii) exactly 1 torch is sold for each of the four consecutive days.

[6]

12. The displacement, in centimetres, of a particle from an origin, O, at time t seconds, is given by $s(t) = 8e^t \sin 3t$, $0 \le t \le \pi$.

(a) Find the maximum distance of the particle from O.

[2]

(b) (i) Find the time when the particle first goes back to O.

(ii) Find $s'(t)$.

(iii) Hence, write down the acceleration of the particle at the instant it first goes back to O.

[5]

13. Two surveys are conducted to measure the residents' satisfaction on the services provided by the community centre. A score from 0 to 10 is used in the surveys, where 0 represents absolute dissatisfaction and 10 represents absolute satisfaction. The table below shows the results of the surveys completed by 6 residents:

Resident	A	B	C	D	E	F
Scores from the first survey (x)	5	7	3	6	8	8
Scores from the second survey (y)	4	9	5	5	9	9

The manager of the community centre wants to investigate whether the mean scores of the second survey has improved. A paired t-test is conducted at a 5% significance level. Define $d = x - y$.

(a) (i) Write down the null hypothesis of the test.

(ii) Write down the alternative hypothesis of the test.

[2]

(b) Find the p-value.

[2]

(c) State the conclusion of the test with a reason.

[2]

14. Let $f(x)=9x+1$, $g(x)=2\sin\left(\frac{x}{3}\right)-6$. Let $h(x)=(g\circ f)(x)$.

(a) Find an expression for $h(x)$.

[3]

(b) Find the period of h.

[2]

(c) Write down the range of h.

[2]

15. A quadratic function is given by $f(x)=ax^2+bx+c$. It is given that the complex roots of $f(x)=0$ are $\frac{1}{2}+\frac{1}{4}\mathrm{i}$ and $\frac{1}{2}-\frac{1}{4}\mathrm{i}$.

(a) Write down the values of

(i) $\left(\frac{1}{2}+\frac{1}{4}\mathrm{i}\right)+\left(\frac{1}{2}-\frac{1}{4}\mathrm{i}\right)$;

(ii) $\left(\frac{1}{2}+\frac{1}{4}\mathrm{i}\right)\left(\frac{1}{2}-\frac{1}{4}\mathrm{i}\right)$.

[2]

(b) Hence, find the expression of $f(x)$, giving the answer in terms of a.

[3]

The graph of $f(x)$ passes through $\left(1, \frac{5}{2}\right)$.

(c) Find the value of a.

[2]

16. The value $V(t)$ of a pendulum clock (in dollars) t years after 31st December, 1888 can be modelled by $V(t)=\begin{cases} \dfrac{1000000}{1+29e^{-2.175}}(t+15) & 0\le t<15 \\ \dfrac{30000000}{1+29e^{-0.145t}} & t\ge 15 \end{cases}$.

(a) Find the value of the pendulum clock at the end of 1899.

[2]

(b) Find t when $V(t)=10000000$.

[2]

(c) Interpret the condition on the value of the pendulum clock after a long period of time.

[1]

17. R is defined to be the region bounded by the lines $y = e^{0.25x} - 1.25$, $y = 8$, the x-axis and the y-axis

(a) (i) Show that $x = 4\ln(y + 1.25)$.

(ii) Hence, find the area of R.

[5]

A solid model is formed by rotating the region R through 2π about the y-axis.

(b) Find the volume of the solid model.

[2]

18. The width of photo frames, in centimetres, sold in a bazaar is studied. 11 photo frames are randomly selected and the corresponding widths are measured. It is given that the sample mean is 38 cm and the width of the 99% confidence interval for the population mean is 13.8 cm.

(a) Explain why the 90% confidence interval for the population mean is a subset of the 99% confidence interval for the population mean.

[1]

(b) Write down the 99% confidence interval for the population mean.

[1]

Let σ^2 be the known population variance.

(c) Find σ^2.

[4]

END OF PAPER

Applications and Interpretation Higher Level for IBDP Mathematics
Practice Paper Set 1 – Paper 2 (120 Minutes)

Question – Answer Book

Instructions

1. Attempt **ALL** questions. Write your answers in the spaces provided in this Question - Answer Book.

2. A graphic display calculator is needed.

3. You are suggested to prepare a formula booklet of Applications and Interpretation for IBDP Mathematics when attempting the questions.

4. Supplementary answer sheets and graph papers will be supplied on request.

5. Unless otherwise specified, **ALL** working must be clearly shown.

6. Unless otherwise specified, numerical answers should be either **EXACT** or correct to **3 SIGNIFICANT FIGURES**.

7. The diagrams in this paper are **NOT** necessarily drawn to scale.

8. Information to be read before you start the exam:

	Marker's Use Only	**Examiner's Use Only**	
Question Number	**Marks**	**Marks**	**Maximum Mark**
1			**17**
2			**17**
3			**19**
4			**12**
5			**15**
6			**17**
7			**13**
Overall			
Paper 2 Total			**110**

1. The equation of the straight line L_1 is given by $3x+y-10=0$. The coordinates of the point P are $(3,1)$.

(a) Show that P lies on L_1.

[1]

(b) Write down the y-intercept of L_1.

[1]

The coordinates of the point Q are $(11,-3)$. M is the mid-point of PQ.

(c) Find

(i) the coordinates of M;

(ii) the gradient of PQ;

(iii) the distance between P and Q.

[6]

The straight line L_2 passes through P and Q.

(d) Show that L_1 and L_2 are not perpendicular.

[2]

The straight line L_3 passes through P and is perpendicular to L_1.

(e) Show that the equation of L_3 is $x-3y=0$.

[4]

L_1 and L_3 intersect with the y-axis at R and S respectively.

(f) Find the area of the triangle PRS.

[3]

2. The relationship between the body temperature and the pulse rate of the students from a sports team is investigated. Six students from the group A of the team are first medically examined and their body temperature and their pulse rates are recorded in the table below.

Student	A	B	C	D	E	F
Body Temperature ($x^\circ\mathrm{C}$)	35.8	36.2	36.4	36.7	37.4	37.1
Pulse Rate (y beats per minute)	80	81	87	117	100	93

(a) The relationship between the variables is modelled by the regression equation $y = ax + b$.

(i) Write down the value of a and of b.

(ii) Hence, estimate the pulse rate of a student whose body temperature is $37^\circ\mathrm{C}$.

[4]

(b) (i) Write down the correlation coefficient.

(ii) State which **two** of the following describe the correlation between the variables.

[3]

positive	strong	zero
negative	weak	moderate

A similar investigation has been completed last year. The pulse rates of 100 students were recorded and the data was presented as follows:

Pulse Rate (y beats per minute)	Frequency
$75 \le y < 85$	16
$85 \le y < 95$	23
$95 \le y < 105$	32
$105 \le y < 115$	12
$115 \le y < 125$	17

Someone claims that the distribution of the data is expected to be evenly distributed. Hence, a χ^2 goodness of fit test is conducted at a 5% significance level.

(c) (i) Write down the null hypothesis of the test.

(ii) Find the p-value.

(iii) Hence, state the conclusion of the test with a reason.

[5]

Another five students from the Group B of the team are also medically examined and their pulse rates are recorded in the table below.

Student	G	H	I	J	K
Pulse Rate (y beats per minute)	95	99	117	87	110

The team manager wants to know whether the mean pulse rates μ_A and μ_B of the students from the Group A and the Group B respectively are different. A t-test is conducted at a 1% significance level.

(d) (i) Write down the alternative hypothesis of the test.

(ii) Find the p-value.

(iii) Hence, state the conclusion of the test with a reason.

[5]

3. The function f is given by $f(x)=\frac{4}{3}x^3+5x^2-6x+2$, $x\in\mathbb{R}$.

(a) Write down the y-intercept of the graph of f.

[1]

(b) Find $f(3)$.

[2]

(c) Find $f'(x)$.

[2]

(d) Solve the equation $f'(x)=0$.

[3]

(e) Write down the equations of the horizontal tangents of the graph of f.

[2]

(f) Write down the range of values of w such that the equation $f(x)=w$ has

(i) three solutions;

(ii) only one solution.

[4]

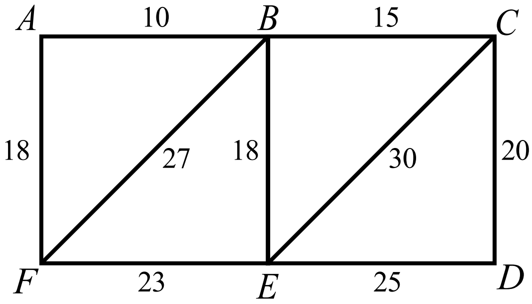
(g) Find the gradient of the tangent at $x=3$.

[2]

(h) Hence, show that the equation of the normal at $x=3$ is $x+60y-3903=0$.

[3]

4. Consider the following weighted graph:



(a) Write down

(i) the degree of B;

(ii) the number of vertices of odd degree;

(iii) the number of vertices of even degree.

[3]

Kruskal's algorithm is used to find the minimum spanning tree for this graph.

(b) State the edge of the smallest weight.

[1]

(c) By using the algorithm, find the minimum spanning tree.

[3]

(d) Write down the weight of the minimum spanning tree.

[1]

(e) Use the Chinese postman algorithm to find a possible route of minimum weight that passes through all edges, starting and finishing at C.

[3]

(f) Write down the corresponding weight of the route.

[1]

5. Let $\mathbf{M} = \begin{pmatrix} 1 & 0.5 \\ 0 & 1 \end{pmatrix}$.

(a) (i) Find $\mathbf{M}^2$.

(ii) Find $\mathbf{M}^3$.

(iii) By using the above results, write down $\mathbf{M}^{30}$.

[5]

Let $s(n) = \mathbf{M} + \mathbf{M}^2 + \mathbf{M}^3 + \cdots + \mathbf{M}^n$, where $n \geq 1$.

(b) (i) Write down $s(2)$.

(ii) Write down $s(3)$.

(iii) By using the above results, find $s(30)$.

[6]

Let $r(n) = \mathbf{M} + \mathbf{M}^2 + \mathbf{M}^4 + \mathbf{M}^8 + \cdots + \mathbf{M}^{2^{n-1}}$, where $n \geq 1$.

(c) Find $r(10)$.

[4]

6. In an experiment, a metal ball moves with velocity $v\,\text{cms}^{-1}$ and displacement $x\,\text{cm}$ with respect to the starting point O. By considering the rate of change of its velocity, the relationship between the variables can be modelled by the differential equation $\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} = 25x$.

(a) By using $v = \dfrac{\mathrm{d}x}{\mathrm{d}t}$, express the differential equation in a coupled system.

[1]

Euler's method with a step length of 0.2 is used to approximate the displacement of the particle at $t = 1$. It is given that initially the particle is at rest with displacement one centimetre.

(b) Find, when $t = 0.2$, the approximate value of

(i) v;

(ii) x.

[4]

(c) Write down the approximate value of the displacement at

(i) $t = 0.4$;

(ii) $t = 1$;

(iii) $t = 2.6$.

[3]

The system can be expressed by a matrix equation $\overline{\dot{\mathbf{X}} = \mathbf{MX}}$, where $\mathbf{M}$ is a 2×2 matrix, and $\dot{\mathbf{X}} = \begin{pmatrix} \dfrac{\mathrm{d}v}{\mathrm{d}t} \\ \dfrac{\mathrm{d}x}{\mathrm{d}t} \end{pmatrix}$ and $\mathbf{X} = \begin{pmatrix} v \\ x \end{pmatrix}$ are two 2×1 matrices. Let λ_1 and λ_2 be the eigenvalues of $\mathbf{M}$, where $\lambda_1 < \lambda_2$.

(d) Find $\det(\mathbf{M} - \lambda\mathbf{I})$, giving the answer in terms of λ.

[2]

(e) Hence, write down the values of λ_1 and λ_2.

[2]

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the eigenvectors of $\mathbf{M}$ corresponding to λ_1 and λ_2 respectively.

(f) Write down $\mathbf{v}_1$ and $\mathbf{v}_2$.

[2]

(g) Hence, show that the particular solution of x is $x = 0.5e^{-5t} + 0.5e^{5t}$.

[3]

7. In a game, two balls A and B are moving in a three-dimensional space, which can be modelled by a three-dimensional coordinate plane. At the start of the game, A is at $(5, 5, 0)$ and its velocity vector is $\begin{pmatrix} -10 \\ 10 \\ 0 \end{pmatrix}$.

(a) Write down the vector equation for the displacement of A, giving the answer in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$,

[2]

After p seconds, A is at $(-85, 95, 0)$.

(b) Find p.

[2]

At the start of the game, B is at $(0, 0, -50)$. After 5 seconds, it is at $(-50, 50, 0)$.

(c) Find the velocity vector of B.

[2]

(d) Write down the vector equation for the displacement of B, giving the answer in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$,

[2]

(e) Find the shortest distance between A and B.

[4]

(f) Hence, write down the time when A and B are closest to each other.

[1]

END OF PAPER

Booklet 1.3

Applications and Interpretation Higher Level for IBDP Mathematics
Practice Paper Set 1 – Paper 3 (60 Minutes)

Question – Answer Book

Instructions

1. Attempt **ALL** questions. Write your answers in the spaces provided in this Question - Answer Book.

2. A graphic display calculator is needed.

3. You are suggested to prepare a formula booklet of Applications and Interpretation for IBDP Mathematics when attempting the questions.

4. Supplementary answer sheets and graph papers will be supplied on request.

5. Unless otherwise specified, **ALL** working must be clearly shown.

6. Unless otherwise specified, numerical answers should be either **EXACT** or correct to **3 SIGNIFICANT FIGURES**.

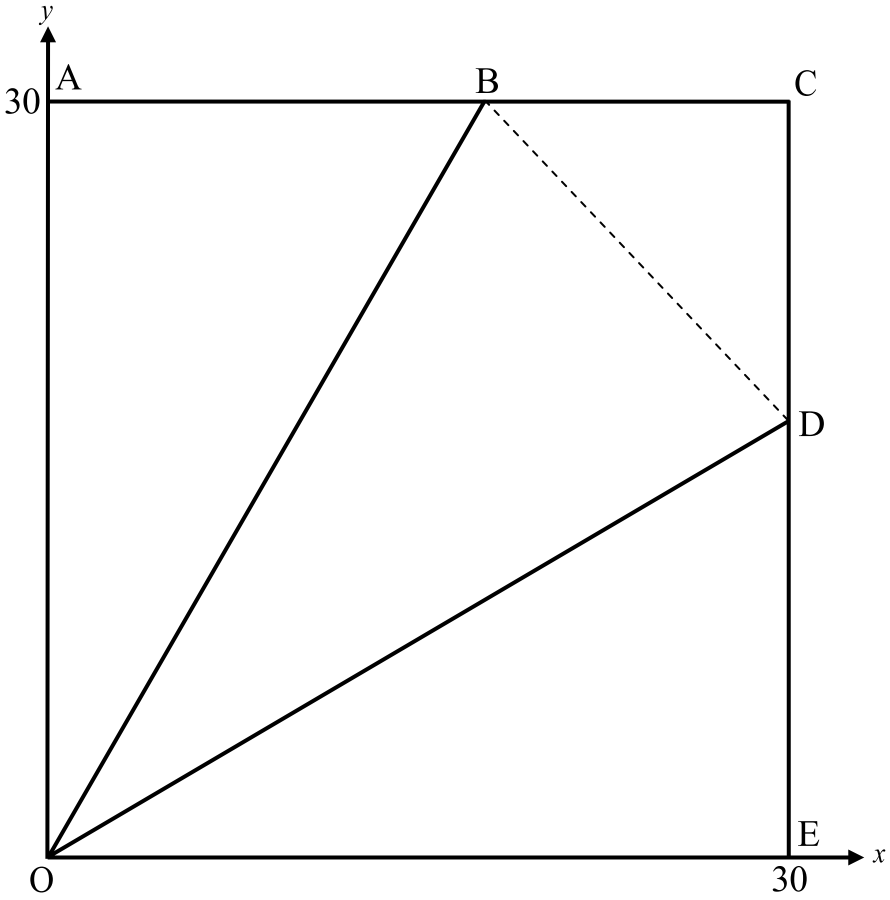
7. The diagrams in this paper are **NOT** necessarily drawn to scale.

8. Information to be read before you start the exam:

	Marker's Use Only	**Examiner's Use Only**	
Question Number	**Marks**	**Marks**	**Maximum Mark**
1			**28**
2			**27**
Overall			
Paper 3 Total			**55**

1. This question aims at investigating the design of a square farm by Voronoi diagrams and graph theory.

The diagram below shows the Voronoi diagram of a square farm OABCDE bounded by the coordinate axes, the lines $x = 30$ and $y = 30$, where 1 unit represents $1\,\mathrm{m}$.



The owner of the farm wishes to create two roads OB and OD to let his car to travel in the farm. He first drafts the directions of the roads such that $\mathrm{A\hat{O}B} = \mathrm{B\hat{O}D} = \mathrm{D\hat{O}E} = \frac{\pi}{6}\,\mathrm{rad}$. It is given that ABC and CDE are straight lines.

Let $\text{area of OAB} : \text{area of OBCD} : \text{area of ODE} = 1 : r : 1$.

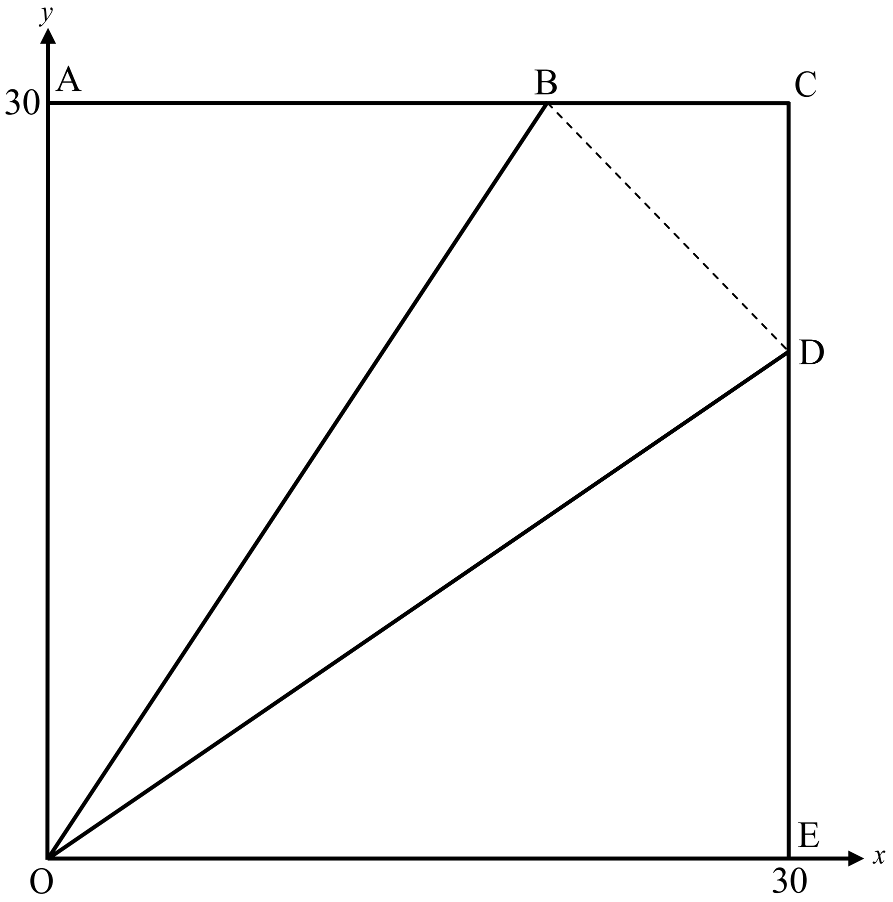
(a) (i) Find DE.

(ii) Show that the area of the triangle ODE is $260\,\mathrm{m}^2$.

(iii) Hence, write down r.

[4]

Consider the case when $r = 1$.



(b) (i) Find DE.

(ii) Hence, find $\mathrm{D\hat{O}E}$ in radians.

(iii) Write down $\mathrm{B\hat{O}D}$ in radians.

[5]

The straight line BD is the boundary of the Voronoi cells of $\mathrm{C}(30, 30)$ and F, where F is the site of the region OBD.

(c) (i) State the geometric relationship between BD and CF.

(ii) Find the coordinates of the mid-point of BD.

(iii) Hence, write down the coordinates of F.

[5]

The owner can walk along OA, ABC, CDE, OE, OB, OD and BD. One scarecrow is placed at each of the six positions O, A, B, C, D and E.

(d) Write down the adjacency matrix $\mathbf{M}$ of the graph.

[3]

(e) Hence, find the total number of walks of length at most 3 from C to E.

[2]

The following table shows the least weight of a path connecting any two vertices.

	O	A	B	C	D	E
O	-	30	36.1	p	36.1	30
A	30	-	20	30	34.1	q
B	36.1	20	-	10	14.1	34.1
C	p	30	10	-	10	30
D	36.1	34.1	14.1	10	-	20
E	30	q	34.1	30	20	-

(f) Write down the value of

(i) p;

(ii) q.

[2]

(g) Using the nearest neighbour algorithm, starting and finishing at O, find an upper bound of the total distance of a cycle that passes through all six positions of scarecrows.

[3]

(h) Using the deleted vertex algorithm by deleting the vertex C, find a lower bound of the total distance of a cycle that passes through all six positions of scarecrows, giving the answer correct to one decimal place.

[4]

2. This question aims at investigating the results of a Mathematics mock examination.

Three hundred students attended a Mathematics mock examination. The following table shows the distribution of their final grades, where 1 represents the lowest grade and 7 represents the highest grade.

Grade	1	2	3	4	5	6	7
Frequency	12	27	58	103	45	35	20

The grades of two students are randomly selected.

(a) (i) Find the probability that both grades are either 5, 6 or 7.

(ii) Given that the probability that both grades are either 5, 6 or 7, find the probability that both grades are the same.

[5]

The organizer of the mock examination claims that 18% of the students attending the mock examination takes the Mathematics Higher level course. A sample of 25 students are interviewed, and 7 of them takes the Mathematics Higher level course.

A hypothesis test is conducted at a 5% significance level to test whether there are actually more than 18% of the students attending the mock examination takes the Mathematics Higher level course.

(b) (i) Write down the null hypothesis of the test.

(ii) Write down the alternative hypothesis of the test.

(iii) Find the p-value.

(iv) Hence, state the conclusion of the test with a reason.

[6]

The following table shows the distribution of the actual scores:

Score (x)	$0 \le x \le 20$	$20 < x \le 40$	$40 < x \le 60$	$60 < x \le 80$	$80 < x \le 100$
Observed Frequency	20	72	140	45	23
Expected Frequency	21.7	77.4	116.7	$84.2 - f$	f

(c) Write down the unbiased estimates for the population

(i) mean;

(ii) standard deviation;

(iii) variance.

[3]

A χ^2 goodness of fit test is conducted at a 5% significance level to determine whether the examination score can be modelled by a normal distribution with parameters values evaluated in (c).

(d) (i) Write down the null hypothesis of the test.

(ii) Write down f.

(iii) Hence, write down the degree of freedom of the test.

(iv) Find the p-value.

(v) Hence, state the conclusion of the test with a reason.

[7]

The organizer starts an online system such that students who are interested in participating in the mock examination in the following month can reserve the quota. The number of students, X, reserving the quota follows a Poisson distribution with parameter λ per hour.

A hypothesis test is conducted at a particular significance level to test whether λ is less than 11.

(e) (i) Write down the null hypothesis of the test.

(ii) Write down the alternative hypothesis of the test.

[2]

The null hypothesis is rejected if it is observed that at most five students reserving the quotas in a particular hour.

(f) Find the probability that a Type I error is made.

[2]

The actually value of λ is 7.

(g) Find the probability that a Type II error is made.

[2]

END OF PAPER

Booklet 2.1

Applications and Interpretation Higher Level for IBDP Mathematics

Practice Paper Set 2 – Paper 1 (120 Minutes)

Question – Answer Book

Instructions

1. Attempt **ALL** questions. Write your answers in the spaces provided in this Question - Answer Book.

2. A graphic display calculator is needed.

3. You are suggested to prepare a formula booklet of Applications and Interpretation for IBDP Mathematics when attempting the questions.

4. Supplementary answer sheets and graph papers will be supplied on request.

5. Unless otherwise specified, **ALL** working must be clearly shown.

6. Unless otherwise specified, numerical answers should be either **EXACT** or correct to **3 SIGNIFICANT FIGURES**.

7. The diagrams in this paper are **NOT** necessarily drawn to scale.

8. Information to be read before you start the exam:

	Marker's Use Only	**Examiner's Use Only**	
Question Number	**Marks**	**Marks**	**Maximum Mark**
1			**6**
2			**6**
3			**6**
4			**5**
5			**6**
6			**6**
7			**6**
8			**6**
9			**7**
10			**7**
11			**7**
12			**6**
13			**5**
14			**8**
15			**5**
16			**6**
17			**7**
18			**5**
Overall			
Paper 1 Total			**110**

1. In January 2021, the number of watermelons grown in each small farm is counted. The following table shows the results.

Number of watermelons	0	1	2	3	4	5
Frequency	12	10	6	5	5	2

(a) Write down

(i) the total number of small farms;

(ii) the median number of watermelons;

(iii) the modal number of watermelons.

[3]

(b) Find the mean number of watermelons.

[2]

(c) State whether the above data is discrete or continuous.

[1]

2. The depth (in metres) of the water in a river on a particular day can be modelled by the function $d = 3.5\sin(3^\circ t) + 6$, where t is the number of minutes after $5:00\ \text{am}$.

(a) Write down

(i) the amplitude of d;

(ii) the maximum value of d;

(iii) the minimum value of d.

[3]

(b) Find the period of d, giving the answer in minutes.

[2]

(c) Write down the time for the third minimum value of d.

[1]

3. The table shows the first four terms of three sequences x_n, y_n and z_n.

n	1	2	3	4
x_n	100	300	500	700
y_n	100	300	400	450
z_n	100	300	900	2700

(a) State which sequence is

(i) arithmetic;

(ii) geometric.

[2]

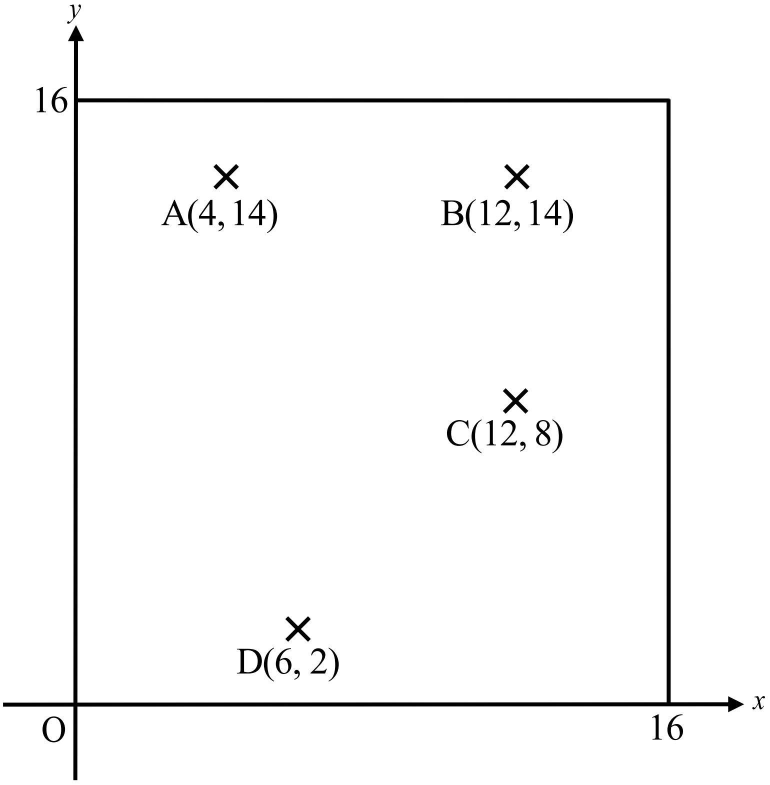
(b) Find the 10th term of the arithmetic sequence.

[2]

(c) Find the sum of the first 10 terms of the geometric sequence.

[2]

4. The diagram below shows the Voronoi diagram of four convenience stores, A, B, C and D, in a city, where 1 unit represents 1 km.



The points $\mathrm{E}(8, 11)$ and $\mathrm{F}\left(\frac{41}{7}, \frac{57}{7}\right)$ are the intersections of the boundaries of Voronoi cells.

Emily is going to start her business by opening a new convenience store at either E or F. She hopes that the location of the new convenience store to be as far as possible from all other convenience stores. Two circles are constructed such that the centres are E and F respectively, and passes through the convenience stores in adjacent Voronoi cells.

(a) Find the radius of the circle centred at

(i) E;

(ii) F.

[4]

(b) Hence, determine which location is farthest from all four convenience stores.

[1]

5. Lucas needs to settle a payment for his postgraduate programme. He is suggested by a bank to choose an option to repay the loan of $\$120000$ with a nominal annual interest rate of 2.95%, compounded monthly:

A monthly payment of $\$2000$ has to be paid
at the end of each month until the loan is fully repaid

(a) Find the number of months to repay the loan, rounding up the answer correct to the nearest month.

[3]

(b) Hence, find the amount of interest paid.

[3]

6. A factory produces jackets. The cost $\$C$ of producing x jackets can be modelled by $C(x)=\frac{1}{2}(x-90)^2+60$.

(a) Find the cost of producing 100 jackets.

[2]

The cost of production should not exceed $\$1310$. To do this the factory needs to produce at least n jackets and at most 140 jackets.

(b) Find the value of n.

[2]

(c) Find the number of jackets produced when the cost of production is lowest.

[2]

7. The height H m of peach trees in a garden are normally distributed with mean 2.85 m and standard deviation 0.19 m.

(a) Write down the probability that the height of a randomly chosen peach tree is

(i) within 1 standard deviation of the mean;

(ii) within 2 standard deviation of the mean.

[2]

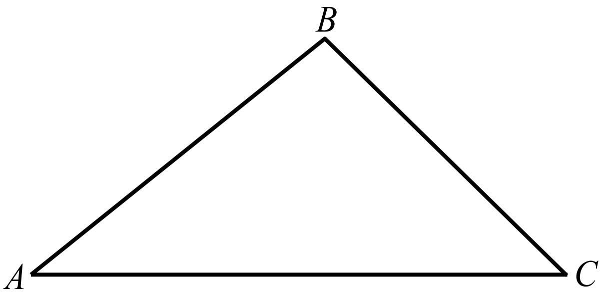
(b) Find the probability that the height of a randomly chosen peach tree is shorter than 2.82 m.

[2]

(c) Find the value of r if 28% of the peach trees are taller than r m.

[2]

8. The following diagram shows a horizontal triangular area ABC reserved for an assembly.



$\mathrm{AB} = 15\,\mathrm{m}$, $\mathrm{BC} = 13.5\,\mathrm{m}$, and $\mathrm{A\hat{B}C} = 98^\circ$.

(a) Find the length of AC.

[3]

(b) Find the size of the angle $\mathrm{B\hat{A}C}$.

[3]

9. The number of goals scored in a randomly selected match by a football team follows a Poisson distribution with mean 3.3.

(a) Find the probability that less than three goals are scored in a particular match.

[2]

(b) Find the probability that exactly ten goals are scored in three randomly selected matches.

[2]

(c) Given that in three randomly selected matches more than nine goals are scored, find the probability that less than fourteen goals are scored.

[3]

10. The relationship between two variables W and x is given by $W = hk^x$, where h, $k \in \mathbb{R}$, $W > 0$.

(a) Express $\ln W$ as a linear function of x.

[3]

A graph of $\ln W$ against x shows a straight line such that the gradient and the vertical intercept of the line are 0.4 and -0.85 respectively.

(b) Find, correct the answers to 5 significant figures, the value of

(i) h;

(ii) k.

[4]

11. The vectors $\mathbf{a}$ and $\mathbf{b}$ are given as $\mathbf{a}=\begin{pmatrix}2\\4\\3\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}-2\\3\\5\end{pmatrix}$.

(a) Write down $3\mathbf{a}+2\mathbf{b}$.

[1]

(b) Hence, find the component of $3\mathbf{a}+2\mathbf{b}$

(i) parallel to $\mathbf{a}$;

(ii) perpendicular to $\mathbf{b}$.

[6]

12. The following table shows the probability distribution of a discrete random variable X.

x	3	5	7	9
$\mathrm{P}(X=x)$	0.3	0.1	0.15	0.45

(a) Find $\mathrm{E}(X)$.

[2]

Another random variable Y is defined such that $\mathrm{E}(Y)=17$ and $\mathrm{Var}(Y)=3$. It is given that X and Y are independent, and $\mathrm{Var}(X)=6.75$.

(b) Find $\mathrm{E}(2X-5Y)$.

[2]

(c) Find $\mathrm{Var}(2X-5Y)$.

[2]

13. The random variable X is defined such that $\mathrm{E}(X) = 700$ and $\mathrm{Var}(X) = 15.5$. A random sample of 320 observations is selected from the distribution of X. Let $\bar{X}$ be the mean of the sample. By using the central limit theorem,

(a) write down $\mathrm{E}(\bar{X})$;

[1]

(b) find $\mathrm{Var}(\bar{X})$;

[2]

(c) find $\mathrm{P}(\bar{X} < 699.83)$.

[2]

14. The population of leopard in a forest, w, t months after 1 April 2020 is given by $w = 237\cos 0.5t + 850$.

(a) Find the number of leopards in the forest on 1 June 2020, correct the answer to the nearest integer.

[3]

(b) Find $\frac{\mathrm{d}w}{\mathrm{d}t}$.

[2]

Assume that there are 30 days per month. After n days, where $n \in \mathbb{Z}$, the rate of increase in w attains its maximum for the first time.

(c) Find the value of n.

[3]

15. The function f is defined as $f(x)=(x+10)^2(x-2)^2$. The domain of f is restricted as $\{x:x\geq k\}$ such that f^{-1} exists.

(a) Write down the coordinates of the local minimum of f which is closest to the origin.

[1]

(b) Hence, write down the minimum value of k.

[1]

f can also be expressed as $f(x)=((x+4)^2-36)^2$.

(c) Find the expression of f^{-1}.

[3]

16. Consider the complex numbers $z_1 = \frac{1}{2}\text{cis}\frac{\pi}{10}$ and $z_2 = \frac{1}{8}\text{cis}\frac{\pi}{4}$.

(a) (i) Express $z_1^{\ 5}$ in the form $r\text{cis}\theta$.

(ii) Hence, write down the real part of $z_1^{\ 5}$.

[3]

(b) Express $\frac{z_1^{\ 5}}{z_2}$ in the form

(i) $r\text{cis}\theta$;

(ii) $r\text{e}^{\text{i}\theta}$.

[3]

17. The relationship between the cost and the revenue of the products in a company is studied. The following table shows the cost x (in USD) of the four products in the company and the corresponding revenue y (in USD).

Cost (x USD)	3	3.5	4	4.5	5
Revenue (y USD)	6	8	8	13	12

It is suggested that the relationship between the variables can be modelled by the regression equation $y = a \cdot b^x$, where a, $b \in \mathbb{R}$.

(a) (i) Write down the least square regression curve for the revenue of products.

(ii) Write down the coefficient of determination, giving the answer in five decimal places.

[3]

(b) Write down SS_{res}, the sum of square residuals.

[2]

(c) By using $R^2 = 1 - \frac{SS_{res}}{SS_{tot}}$, find SS_{tot}, the total sum of squares.

[2]

18. Let x and y be the number of soldiers, in thousands, from country X and country Y respectively. The changes in the number of soldiers in a battle between the two countries can be modelled by the coupled differential equations $\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = (13-2y)x \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = (4x-16)y \end{cases}$.

(a) When $y > 0$, state the range of values of x such that $\frac{\mathrm{d}y}{\mathrm{d}t} > 0$.

[1]

The initial numbers of soldiers from the two countries are both 4500. Euler's method with a step length of 0.05 is used to approximate the numbers of soldiers from the two countries when $t = 0.05$.

(b) Find the approximate numbers of soldiers from country Y at $t = 0.05$.

[4]

END OF PAPER

Applications and Interpretation Higher Level for IBDP Mathematics
Practice Paper Set 2 – Paper 2 (120 Minutes)

Question – Answer Book

Instructions

1. Attempt **ALL** questions. Write your answers in the spaces provided in this Question - Answer Book.

2. A graphic display calculator is needed.

3. You are suggested to prepare a formula booklet of Applications and Interpretation for IBDP Mathematics when attempting the questions.

4. Supplementary answer sheets and graph papers will be supplied on request.

5. Unless otherwise specified, **ALL** working must be clearly shown.

6. Unless otherwise specified, numerical answers should be either **EXACT** or correct to **3 SIGNIFICANT FIGURES**.

7. The diagrams in this paper are **NOT** necessarily drawn to scale.

8. Information to be read before you start the exam:

	Marker's Use Only	**Examiner's Use Only**	
Question Number	**Marks**	**Marks**	**Maximum Mark**
1			**20**
2			**15**
3			**18**
4			**17**
5			**13**
6			**14**
7			**13**
Overall			
Paper 2 Total			**110**

1. Sardar owns a plot of land in the shape of a pentagon ABCDE, where the boundaries AE and BC are parallel. The dimension can be represented on a coordinate plane, where 1 unit on the plane represents $1\,\mathrm{m}$. The coordinates of A, B, C and E are $\mathrm{A}(10,10)$, $\mathrm{B}(10,110)$, $\mathrm{C}(50,110)$ and $\mathrm{E}(125,10)$ respectively. The equation of the boundary CD is $7x+24y-2990=0$.

(a) It is given that $\mathrm{F}(98,f)$ lies on CD. Find f.

[2]

(b) Write down the gradient of CD.

[1]

The boundaries CD and DE are perpendicular to each other.

(c) (i) Find the gradient of DE.

(ii) Hence, show that the equation of DE is $24x-7y-2930=0$.

[5]

(d) Write down the coordinates of D, the point of intersection between the boundaries CD and DE.

[2]

(e) Show that F is the mid-point of CD.

[2]

(f) Find the length of the boundary DE.

[2]

It is also given that the length of the boundary CD is $100\,\mathrm{m}$.

(g) Find the area of the triangle CDE.

[2]

(h) Hence, find the total area of the plot of land.

[4]

2. In a library, the habit of reading books of young people is investigated. The table below shows the number of fiction books read by two different groups of people in a particular month, after they have filled in a survey:

The group of 10-year-old students	3	4	7	6	9	11	3	1	4	6
The group of 20-year-old people	4	6	5	4	0	1	0	3	2	

The library manager wants to investigate whether 10-year-old students read more than 20-year-old people on average. Let μ_1 and μ_2 be the mean number of fiction books read by the 10-year-old students and the 20-year-old people respectively. A t-test is conducted at a 5% significance level, and it is assumed that the variances of the number of fiction books read by two different groups of people are the same.

(a) Write down the alternative hypothesis of the test.

[1]

(b) Find the p-value.

[2]

(c) State the conclusion of the test with a reason.

[2]

One person from each age group above is randomly selected.

(d) (i) Find the probability that both of them read at least five fiction books.

(ii) Find the probability that at least one of them read at least five fiction books.

[4]

The library manager also wants to investigate the relationship between the ages and the reading preferences of 120 library users from another three different age groups. A survey had been conducted and the following table shows the results:

		Reading preference		
		Fiction	Novel	Newspaper
Age	30-year-old	11	25	4
	40-year-old	16	16	8
	50-year-old	5	3	32

A χ^2 test for independence is conducted at a 1% significance level.

(e) Write down the alternative hypothesis of the test.

[1]

(f) Write down the degree of freedom of the test.

[1]

(g) Find the value of χ^2_{calc}, the test statistic.

[2]

The critical value is given by 13.277.

(h) State the conclusion of the test with a reason.

[2]

3. The function f is given by $f(x)=-x^3+bx^2-432x+2456$, b, $x\in\mathbb{R}$. It is given that the local minimum point is located at $x=8$.

(a) Find b.

[4]

(b) Hence, write down

(i) the y-coordinate of the local minimum;

(ii) the coordinates of the local maximum.

[3]

(c) Write down the range of values of x such that f is increasing.

[2]

(d) Write down the range of values of k such that the equation $f(x)=k$ has

(i) three solutions;

(ii) at most two solutions.

[4]

The average cost $C(x)$ dollars of producing x thousands smart watches in a company can be modelled by the function $C(x)=-x^3+bx^2-432x+2456$, $0\le x\le 25$.

(e) Show that the average cost attains its minimum when 25000 smart watches are produced.

[2]

(f) Find the range of values of x such that the average cost is not greater than $\$984$.

[3]

4. A particle moves in a straight line. Its velocity, $v \text{ ms}^{-1}$, at time t seconds, is given by

$$v(t) = -0.5(t-5)^3 \text{, for } 0 \le t \le 10 .$$

(a) Find the initial velocity.

[2]

(b) Find the value of t for which the velocity of the particle is -13.5 ms^{-1}.

[3]

(c) Find the total distance the particle travels during the first ten seconds.

[3]

(d) Find the expression of $a(t)$, the acceleration of the particle.

[2]

(e) Find all possible values of t for which the velocity and acceleration are both non-negative.

[3]

It is given that the initial displacement of the particle is -78 m.

(f) Find the expression of $s(t)$, the displacement of the particle.

[4]

5. The vector equations of the lines L_1 and L_2 are $\mathbf{r} = \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix} + t\begin{pmatrix} 2 \\ -6 \\ -2 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + s\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$ respectively. Let A and B be the points on L_1 and L_2 with parameters $t = 5$ and $s = 5$ respectively.

(a) Show that the coordinates of C, the point of intersection of L_1 and L_2, are $(7, -6, 5)$.

[4]

(b) Find the acute angle between L_2 and the z-axis, giving the answer in radians.

[3]

(c) (i) Write down $\overrightarrow{\mathrm{CA}}$.

(ii) Write down $\overrightarrow{\mathrm{CB}}$.

(iii) Hence, find the area of the triangle ABC.

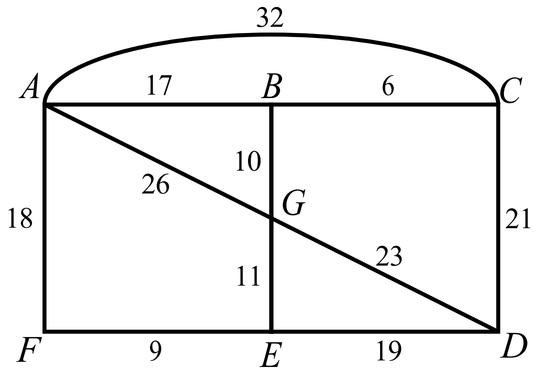
[5]

D is a point such that $\overrightarrow{\mathrm{BD}} = \overrightarrow{\mathrm{CA}}$.

(d) Write down the area of the quadrilateral BDAC.

[1]

6. Consider the following weighted graph:



(a) Is Eulerian circuit exists in the above graph? Explain your answer.

[2]

Prim's algorithm, starting at C, is used to find the minimum spanning tree for this graph.

(b) State the edge of the least weight.

[1]

(c) By using the algorithm, find the minimum spanning tree.

[3]

(d) Write down the weight of the minimum spanning tree.

[1]

The following table shows the least weight of a path connecting any two vertices.

	A	B	C	D	E	F	G
A	-	17	23	44	27	18	26
B	17	-	6	27	21	30	10
C	23	6	-	21	27	36	16
D	44	27	21	-	19	28	23
E	27	21	27	19	-	9	11
F	18	30	36	28	9	-	20
G	26	10	16	23	11	20	-

(e) Using the nearest neighbour algorithm, starting and finishing at G, find an upper bound of the total weight of a cycle that passes through all seven vertices.

[3]

(f) Using the deleted vertex algorithm by deleting the vertex G, find a lower bound of the total weight of a cycle that passes through all seven vertices.

[4]

7. The matrix $\mathbf{M}$ is defined by $\mathbf{M} = \begin{pmatrix} \frac{5}{3} & \frac{4}{3} \\ -\frac{2}{3} & -\frac{1}{3} \end{pmatrix}$. Let λ_1 and λ_2 be the eigenvalues of $\mathbf{M}$, where $\lambda_1 < \lambda_2$.

(a) Find the characteristic polynomial of $\mathbf{M}$.

[2]

(b) Hence, write down the values of λ_1 and λ_2.

[2]

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the eigenvectors of $\mathbf{M}$ corresponding to λ_1 and λ_2 respectively.

(c) Write down $\mathbf{v}_1$ and $\mathbf{v}_2$.

[2]

It is given that $\mathbf{M}^n = \mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}$, where $\mathbf{P}$ is a 2×2 matrix and $\mathbf{D}$ is a 2×2 diagonal matrix.

(d) Write down

(i) $\mathbf{P}$;

(ii) $\mathbf{D}^n$.

[3]

(e) Hence, express $\mathbf{M}^n$ in terms of n.

[3]

Let $g(n)$ be the first diagonal entry of $\mathbf{M}^n$.

(f) Write down $\lim_{n\to\infty} g(n)$.

[1]

END OF PAPER

Applications and Interpretation Higher Level for IBDP Mathematics
Practice Paper Set 2 – Paper 3 (60 Minutes)

Question – Answer Book

Instructions

1. Attempt **ALL** questions. Write your answers in the spaces provided in this Question - Answer Book.
2. A graphic display calculator is needed.
3. You are suggested to prepare a formula booklet of Applications and Interpretation for IBDP Mathematics when attempting the questions.
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5. Unless otherwise specified, **ALL** working must be clearly shown.
6. Unless otherwise specified, numerical answers should be either **EXACT** or correct to **3 SIGNIFICANT FIGURES**.
7. The diagrams in this paper are **NOT** necessarily drawn to scale.
8. Information to be read before you start the exam:

	Marker's Use Only	**Examiner's Use Only**	
Question Number	**Marks**	**Marks**	**Maximum Mark**
1			**30**
2			**25**
Overall			
Paper 3 Total			**55**

1. This question aims at investigating a coin tossing game by probability models and graph theory.

A coin tossing game consists of a coin with two faces: head and tail, where the probability of getting a head is p. The player of this game keeps tossing the coin and the results are recorded. The game stops if the player gets three consecutive heads. If a tail is shown, the player will restart the game.

Let $p = \frac{1}{2}$. Four states are used to describe the game:

A: The player starts the game
B: One head has shown
C: Two consecutive heads have shown
D: Three consecutive heads have shown

(a) Sketch a directed graph to represent the flow of this game.

[4]

(b) State the geometric meaning of the column sum of a particular column in an adjacency matrix of a graph.

[1]

(c) Write down

(i) the adjacency matrix $\mathbf{M}$ of the graph;

(ii) the transition matrix $\mathbf{T}$ of the graph.

[4]

Let $\mathbf{v}_n$ be the state probability vector after the coin is tossed for n times.

(d) (i) Explain why $\mathbf{v}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

(ii) Hence, write down $\mathbf{v}_1$ and $\mathbf{v}_2$.

It is given that $\mathbf{v}_3 = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix}$.

(iii) Explain why $\alpha_1 = \frac{1}{2}$.

(iv) Write down $\alpha_2 : \alpha_3 : \alpha_4$.

[7]

(e) Show that the steady state probability vector is $\mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

[4]

Now, the fair coin used in the game is replaced by another biased coin such that $p = \frac{1}{3}$.

(f) Find the probability that a player tosses the coin for

(i) five times to get the first three consecutive heads;

(ii) seven times to get the first three consecutive heads;

(iii) at least eight times.

[10]

2. This question aims at investigating the weights of a population of crabs.

Alex owns a crab farm in Vietnam. During an interview he claims that the weights W of the crabs in his crab farm is normally distributed with mean 300 grams and standard deviation 10 grams.

(a) (i) Write down the probability that a randomly selected crab from the crab farm weighs less than 292 grams.

A sample of a dozen of crabs is randomly selected.

(ii) Find the probability that the sample mean weight of this sample is less than 292 grams.

[4]

A crab cage is designed such that the weight limit of the cage is 6.1 kilograms. Alex is trying to carry twenty crabs by the cage.

(b) (i) Write down the expected total weight of the twenty crabs.

(ii) Find the variance of the total weight of the twenty crabs.

(iii) Hence, find the probability that the total weight exceeds the weight limit of the cage.

[5]

Alex would like to conduct a hypothesis test at a 5% significance level to test whether there is a negative correlation between the weight of a crab with its maximum walking speed. The table below shows the data of a sample of five crabs:

Weight (W g)	280	286	292	298	304
Maximum walking speed (V cm/s)	86	66	66	48	49

It is also assumed that the maximum walking speeds of crabs are normally distributed. Let ρ be the product moment correlation coefficient of the two variables.

(c) (i) Write down the null hypothesis of the test.

(ii) Write down the alternative hypothesis of the test.

(iii) Find the p-value.

(iv) Hence, state the conclusion of the test with a reason.

[6]

This data can be modelled by the regression line with equation $V = aW + b$.

(d) (i) Write down the value of a and of b.

(ii) Explain what the gradient a represents.

[3]

The reporter in the interview would like to verify whether Alex's claim about the mean weight of the crabs is correct. Later, fifteen crabs are randomly selected to form a sample such that the sample mean weight is 297 grams. It is given that the population standard deviation is 10 grams in the hypothesis test conducted at a 10% significance level.

(e) (i) Write down the null hypothesis of the test.

(ii) Write down the alternative hypothesis of the test.

(iii) Write down the test statistic z.

(iv) Find the p-value.

(v) State the conclusion of the test with a reason.

[7]

END OF PAPER

Applications and Interpretation Higher Level for IBDP Mathematics
Practice Paper Set 3 – Paper 1 (120 Minutes)

Question – Answer Book

Instructions

1. Attempt **ALL** questions. Write your answers in the spaces provided in this Question - Answer Book.
2. A graphic display calculator is needed.
3. You are suggested to prepare a formula booklet of Applications and Interpretation for IBDP Mathematics when attempting the questions.
4. Supplementary answer sheets and graph papers will be supplied on request.
5. Unless otherwise specified, **ALL** working must be clearly shown.
6. Unless otherwise specified, numerical answers should be either **EXACT** or correct to **3 SIGNIFICANT FIGURES**.
7. The diagrams in this paper are **NOT** necessarily drawn to scale.
8. Information to be read before you start the exam:

	Marker's Use Only	**Examiner's Use Only**	
Question Number	**Marks**	**Marks**	**Maximum Mark**
1			**6**
2			**5**
3			**6**
4			**6**
5			**6**
6			**6**
7			**6**
8			**5**
9			**8**
10			**7**
11			**6**
12			**6**
13			**6**
14			**6**
15			**5**
16			**6**
17			**8**
18			**6**
Overall			
Paper 1 Total			**110**

1. The 11th term, u_{11}, and the 31st term, u_{31}, of an arithmetic sequence are 100 and 260 respectively.

(a) Find the common difference of the sequence. [2]

(b) Find the first term of the sequence. [2]

(c) Find the sum of the first 51 terms of the sequence. [2]

2. The number of chocolate eggs eaten by a group of 12 children on a particular day are as follows:

4 1 6 7 2 6 5 7 4 4 3 2

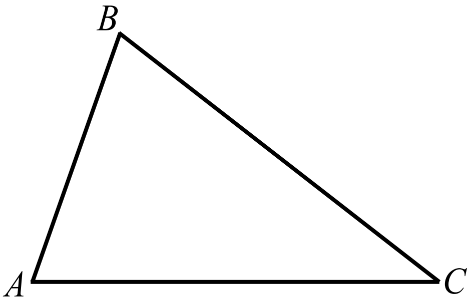
(a) Write down the mode.
[1]

(b) Find the inter-quartile range.
[2]

A children from the group is randomly selected.

(c) Find the probability that the child selected ate at most 5 chocolate eggs.
[2]

3. The following diagram shows triangle ABC.



AB = 35 cm, AC = BC = 54 cm

(a) Find the size of the smallest interior angle of the triangle ABC.

[3]

(b) Hence, find the area of the triangle ABC.

[3]

4. A discrete random variable X has the following probability distribution.

x	1	2	3	4
$\mathrm{P}(X=x)$	$5k^2$	k^2+6k	k^2+k	k^2

(a) Show that $k=\frac{1}{8}$.

[3]

(b) Find $\mathrm{P}(X=2 \mid X\leq 2)$.

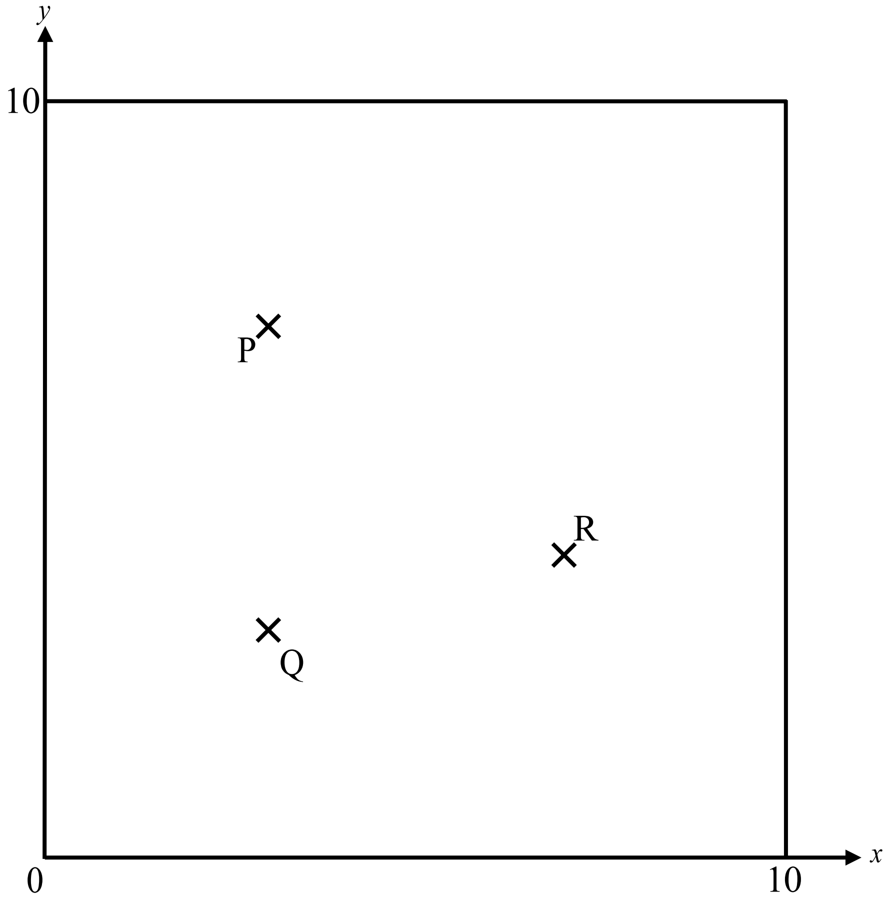
[3]

5. The Voronoi diagram of two points $\mathrm{P}(3, 7)$ and $\mathrm{Q}(3, 3)$ are formed.

(a) Write down the equation of the boundary separating the Voronoi cells of P and Q.

[1]

A new point $\mathrm{R}(7, 4)$ is added to this Voronoi diagram, as shown in the following diagram.



The equation of L, the boundary separating the Voronoi cells of Q and R, is $kx+2y-47=0$, where k is a constant.

(b) (i) Write down the coordinates of the mid-point of QR.

(ii) Find the value of k.

(iii) Hence, find the coordinates of the intersection of the boundaries.

[5]

6. Consider the graph of the function $f(x)=\dfrac{1-8x}{2-7x}$, $x\neq\dfrac{2}{7}$.

(a) Write down the equation of the horizontal asymptote.

[2]

(b) Write down the range of $f(x)$.

[1]

(c) Let $g(x)=\dfrac{1}{2}x^2$. Find the range of values of x such that $f(x)>g(x)$.

[3]

7. Teddy invested 89000 USD in an account that pays a nominal annual interest rate of 9%, compounded monthly. This amount is invested for 6 years and the inflation rate in these 6 years is $i\%$.

(a) Find the real interest rate per year, giving the answer in terms of i and correct to 5 decimal places.

[3]

It is given that the real value of amount of money after 6 years is 118000 USD.

(b) Using the answer in (a), find the value of i.

[3]

8. A factory packages melon juice in cylindrical containers with a base radius of 4 cm and a height of 15 cm.

(a) Find the volume of one container, giving the answer in terms of π.

[2]

(b) Find the total surface area of the container, giving the answer in terms of π.

[2]

(c) Write down the least number of the cylindrical containers needed such that the sum of the total surface areas of the containers is at least 12000 cm^2.

[1]

9. Let $f(x)=30+9x^2+2x^3$.

(a) Find $f'(x)$.

[2]

(b) Hence, solve the equation $f'(x)=0$.

[2]

(c) (i) Write down $f''(x)$.

(ii) Hence, determine the x-coordinate of the local maximum of f.

(iii) Write down the y-coordinate of the local maximum of f.

[4]

10. The following table shows the number of accidents on a road junction in the last 200 days:

Number of accidents	0	1	2	3	4	5 or more
Number of days	16	41	60	43	19	21

A χ^2 goodness of fit test is conducted at a 5% significance level to determine whether the data can be modelled by Poisson distribution with mean 3.

(a) Write down the null hypothesis of the test.

[1]

(b) Write down the value of the expected frequency of the category "5 or more".

[1]

(c) Write down the degree of freedom of the test.

[1]

(d) Find the value of χ^2_{calc}, the test statistic.

[2]

The critical value is given by 11.070.

(e) State the conclusion of the test with a reason.

[2]

11. Find the value of each of the following, giving your answer as an integer.

(a) $\log\frac{1}{8}+\log\frac{1}{125}$

[3]

(b) $\ln e^{\frac{10}{3}}-\ln\sqrt[6]{e}$

[3]

12. The weight of a wooden block, in grams, selected from a factory is studied. A sample of 12 observations is recorded as below:

700, 698, 692, 685, 637, 671, 654, 651, 695, 608, 624, 641

(a) Find an unbiased estimate of the population mean.

[2]

(b) Find s_{n-1}.

[2]

(c) Construct a 99% confidence interval for the population mean, giving the answer to two decimal places.

[2]

13. A quadratic function f is given by $f(x)=(x+1)^2+3$.

Let $g(x)$ be another quadratic function. The graph of g is obtained by a reflection of the graph of f in the x-axis.

(a) Find $g(x)$, giving the answer in the form $a(x+m)^2+n$.

[2]

Let $h(x)=-(x-10)^2$. The graph of h is obtained from g by a translation of $\begin{pmatrix} p \\ q \end{pmatrix}$.

(b) Find the values of

(i) p;

(ii) q.

[4]

14. The number of delays of tram services in a day follows a Poisson distribution with mean 1.75.

(a) Find the probability that there are at least 3 delays in a day.

[2]

(b) Find the probability that there are at most 14 delays in a week.

[2]

(c) Find the probability that in 7 consecutive days there are at most 2 delays on each day.

[2]

15. Consider the graph of $f(y)=-y^3+17y^2-86y+112$.

(a) Find the y-intercept(s) of $f(y)$.

[3]

(b) Hence, find the total area of the regions enclosed by the graph of f and the y-axis.

[2]

16. The following table shows the lengths x (in cm) of the forearm of three students and the corresponding heights y (in cm).

Forearm length $(x \text{ cm})$	24	26	28
Height $(y \text{ cm})$	160	160	173

Two different models are suggested to model the relationship between the variables. The following table shows the estimated heights under two models:

	Height $(y \text{ cm})$ estimated by	
	Model 1	Model 2
Forearm length $(x \text{ cm})$	$y=1.6x^2-81x+1175$	$y=33\sqrt{x}$
24	a	$33\sqrt{24}$
26	b	c
28	161.4	$33\sqrt{28}$

(a) Write down the values of

(i) a;

(ii) b;

(iii) c,

giving the answers correct to 1 decimal place.

[3]

(b) Hence, calculate SS_{res}, the sum of square residuals of the Model 2.

[2]

The model with the smaller sum of square residuals is selected. It is given that the sum of square residuals of the Model 1 is 277.68.

(c) Determine which model is selected.

[1]

17. Let $\mathbf{A}=\begin{pmatrix} 2 & q & 1 \\ p & -3 & -5 \\ -1 & 1 & 3 \end{pmatrix}$, where p, $q \in \mathbb{Z}$. It is given that

$$\mathbf{A}^{-1}=\begin{pmatrix} 0.4 & 0.2 & 0.2 \\ 0.1 & -0.7 & -1.2 \\ 0.1 & 0.3 & 0.8 \end{pmatrix}.$$

(a) Find p and q.

[3]

(b) Hence, solve the system $\begin{cases} 4x+2y+2z=3 \\ x-7y-12z=5 \\ x+3y+8z=9 \end{cases}$.

[5]

18. (a) Solve $\sin 5x + \cos 4x = 0$, for $0 \le x \le \frac{\pi}{2}$.

[2]

(b) Is it true that there are four distinct real roots for the equation $\sin 10x + \cos 8x = 0$, for $0 \le x \le \frac{\pi}{4}$? Using (a) to explain your answer.

[3]

(c) Write down the number of solution(s) of $\sin 5x + \cos 4x = 1$ for $-6 \le x \le 0$.

[1]

END OF PAPER

Booklet 3.2

Applications and Interpretation Higher Level for IBDP Mathematics
Practice Paper Set 3 – Paper 2 (120 Minutes)

Question – Answer Book

Instructions

1. Attempt **ALL** questions. Write your answers in the spaces provided in this Question - Answer Book.
2. A graphic display calculator is needed.
3. You are suggested to prepare a formula booklet of Applications and Interpretation for IBDP Mathematics when attempting the questions.
4. Supplementary answer sheets and graph papers will be supplied on request.
5. Unless otherwise specified, **ALL** working must be clearly shown.
6. Unless otherwise specified, numerical answers should be either **EXACT** or correct to **3 SIGNIFICANT FIGURES**.
7. The diagrams in this paper are **NOT** necessarily drawn to scale.
8. Information to be read before you start the exam:

	Marker's Use Only	**Examiner's Use Only**	
Question Number	**Marks**	**Marks**	**Maximum Mark**
1			**16**
2			**18**
3			**18**
4			**16**
5			**16**
6			**14**
7			**12**
Overall			
Paper 2 Total			**110**

1. Five metal ingots were chosen at random and measurements were made of their breaking strength s and their hardness h. The results are shown in the table below.

Breaking strength (s tonnes per cm)	5	5.5	6	6.5	7
Hardness (h)	60	68	71	70	73

The relationship between the variables can be modelled by the regression equation $h = as + b$.

(a) Write down the value of a and of b.

[2]

(b) Use the regression equation to estimate the hardness of a metal ingot when its breaking strength is 6.3 tonnes per cm.

[2]

There are 120 metal ingots in total. The cumulative frequency below shows the distribution of the hardness of all metal ingots.

Hardness less than	Cumulative Frequency
55	0
60	30
65	56
70	90
75	108
80	120

A metal ingot is randomly selected.

(c) Find the probability that the hardness of the selected ingot is at least 65.

[2]

Ten metal ingots are randomly selected with replacement.

(d) (i) Find the probability that exactly five selected ingots of the hardness at least 65 are selected.

(ii) Find the probability that less than four selected ingots of the hardness at least 65 are selected.

(iii) Write down the expected number of metal ingots of the hardness at least 65.

[5]

Later, another seven metal ingots were chosen at random and measurements were made of their hardness h. The results are shown in the table below.

Hardness (h)	67	82	79	63	90	80	83

A t-test is conducted at a 5% significance level to investigate whether the means of hardness μ_1 and μ_2 from the two samples are different.

(e) (i) Write down the alternative hypothesis of the test.

(ii) Find the p-value.

(iii) Hence, state the conclusion of the test with a reason.

[5]

2. The number of elephants P in a small jungle is modelled by $P(t) = a + b \times c^{-t}$, where t is the number of years after 1st January, 2019, and a, b and c are constants.

It is given that there are 116 elephants on 1st January, 2019.

(a) Find an equation connecting a and b.

[2]

There are 172 elephants on 1st January, 2020.

(b) Find the another equation connecting a, b and c.

[2]

c satisfies the equation $\log_c 81 = 4$.

(c) (i) Show that $c = 3$.

(ii) Hence, find a and b.

[5]

(d) Find the number of elephants on 1st January, 2022, giving the answer correct to the nearest integer.

[2]

(e) Write down the minimum value of the upper bound of the number of elephants.

[1]

(f) Find the number of years needed after 1st January, 2019 when the number of elephants first exceeds 195.

[2]

(g) Someone claims that the amount of time when the number of elephants is between 170 and 180 is the half of that when the number of elephants is between 180 and 190. Is the claim correct? Explain your answer.

[4]

3. The function f is defined as $f(x) = -0.25x^2 + 2x + 4$, $0 \le x \le 8$.

(a) Write down

(i) the coordinates of the vertex;

(ii) the range of $f(x)$.

[4]

(b) Find $f'(x)$.

[2]

P is a point of the graph of f such that the slope of tangent to the graph of f at P is -1.

(c) Show that the coordinates of P are $(6, 7)$.

[4]

(d) Hence, find the equation of the tangent to the graph of f at P, giving the answer in the form $Ax + By + C = 0$.

[2]

The following table shows the functional values of $f(x)$ from $x = 0$ to $x = 8$:

x	0	1	2	3	4	5	6	7	8
$f(x)$	$f(0)$	$f(1)$	7	7.75	8	7.75	7	5.75	4

(e) Write down

(i) $f(0)$;

(ii) $f(1)$.

[2]

(f) Hence, use the trapezoidal rule with 8 intervals, find an estimate for $\int_0^8 f(x)\mathrm{d}x$.

[3]

It is given that the exact value of $\int_0^8 f(x)\mathrm{d}x$ is $\frac{160}{3}$.

(g) State whether the estimate in (f) overestimates or underestimates $\int_0^8 f(x)\mathrm{d}x$.

[1]

4. Two waves are given as $W_1 = 11\cos(2\pi t - 0.1)$ and $W_2 = 13\cos(2\pi t - 0.3)$ respectively, where W_1 and W_2 represent the amplitudes of the two waves respectively. t represents time in seconds. The total amplitude W is given by $W = W_1 + W_2$.

(a) Find the period of W_2.

[2]

It is given that $W_1 + W_2 = \mathrm{Re}(e^{2\pi t \mathrm{i}}(z+w))$, z, $w \in \mathbb{C}$. w is in the form $re^{-0.3\mathrm{i}}$.

(b) Find the expression of $z+w$.

[3]

(c) Express the following in the form $r(\cos\theta + i\sin\theta)$:

(i) z

(ii) w

[2]

(d) It is given that $z+w=Le^{\mathrm{i}\alpha}$. Find

(i) L;

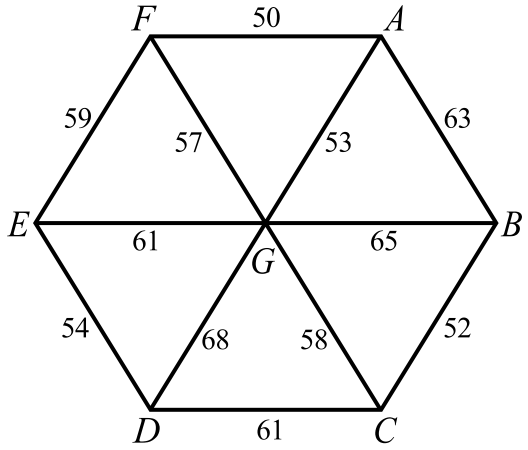
(ii) α.

[6]

(e) Using $W_1 + W_2 = \mathrm{Re}(e^{2\pi t \mathrm{i}}(z+w))$, express W in the form $A\cos(Bt+C)$, A, B, $C \in \mathbb{R}$.

[3]

5. Consider the following weighted graph:



(a) Is Eulerian trail exists in the above graph? Explain your answer. [2]

(b) Write down the adjacency matrix $\mathbf{M}$ of the graph. [2]

(c) Hence, find the total number of walks of length 4 from D to A. [2]

Kruskal's algorithm is used to find the minimum spanning tree for this graph.

(d) By using the algorithm, find the minimum spanning tree. [3]

(e) Write down the weight of the minimum spanning tree. [1]

The following table shows the least weight of a path connecting any two vertices.

	A	B	C	D	E	F	G
A	-	63	111	121	109	50	53
B	63	-	52	113	126	113	65
C	111	52	-	61	115	115	58
D	121	113	61	-	54	113	68
E	109	126	115	54	-	59	61
F	50	113	115	113	59	-	57
G	53	65	58	68	61	57	-

(f) Using the nearest neighbour algorithm, starting and finishing at E, show that an upper bound of the total weight of a cycle that passes through all seven vertices is 398. [2]

(g) Using the deleted vertex algorithm by deleting the vertex E, find a lower bound of the total weight of a cycle that passes through all seven vertices.

[4]

6. The coordinates of the points O, A, B, C and D are $(0,0,0)$, $(0,0,\pi)$, $(\pi,0,\pi)$, $(\pi,0,0)$ and $(0,-\pi,0)$ respectively. E is a point on the line segment BD such that $\mathrm{CE}\perp\mathrm{BD}$.

(a) (i) Find the vector equation of BD, giving the answer in the form $\mathbf{r}=\mathbf{a}+t\mathbf{b}$.

(ii) Express $\overrightarrow{\mathrm{CE}}$ in terms of t.

(iii) Hence, show that the coordinates of E are $\left(\frac{2}{3}\pi,-\frac{1}{3}\pi,\frac{2}{3}\pi\right)$.

[7]

(b) (i) Write down $\overrightarrow{\mathrm{BA}}$.

Let $\mathbf{w}=\overrightarrow{\mathrm{BA}}\times\overrightarrow{\mathrm{BD}}$.

(ii) Find $\mathbf{w}$.

(iii) Hence, find the acute angle between $\mathbf{w}$ and $\begin{pmatrix}1\\1\\2\end{pmatrix}$.

[7]

7. A particle moves in a straight line with velocity $v\ \mathrm{ms^{-1}}$ and displacement $x\ \mathrm{m}$ with respect to the starting point O. By considering the rate of change of its velocity, the relationship between the variables can be modelled by the differential equation $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}-7\frac{\mathrm{d}x}{\mathrm{d}t}+10x=0$.

(a) By using $v=\frac{\mathrm{d}x}{\mathrm{d}t}$, express the differential equation in a coupled system.

[1]

The system can be expressed by a matrix equation $\dot{\mathbf{X}}=\mathbf{MX}$, where $\mathbf{M}$ is a 2×2 matrix, and $\dot{\mathbf{X}}=\begin{pmatrix}\frac{\mathrm{d}v}{\mathrm{d}t}\\ \frac{\mathrm{d}x}{\mathrm{d}t}\end{pmatrix}$ and $\mathbf{X}=\begin{pmatrix}v\\ x\end{pmatrix}$ are two 2×1 matrices. Let λ_1 and λ_2 be the eigenvalues of $\mathbf{M}$, where $\lambda_1<\lambda_2$.

(b) Find $\det(\mathbf{M}-\lambda\mathbf{I})$, giving the answer in terms of λ.

[2]

(c) Hence, write down the values of λ_1 and λ_2.

[2]

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the eigenvectors of $\mathbf{M}$ corresponding to λ_1 and λ_2 respectively.

(d) Write down $\mathbf{v}_1$ and $\mathbf{v}_2$.

[2]

Initially, the particle is at O and its velocity is $3\ \mathrm{ms^{-1}}$.

(e) Find the particular solutions of the velocity and the displacement of the particle.

[5]

END OF PAPER

Applications and Interpretation Higher Level for IBDP Mathematics
Practice Paper Set 3 – Paper 3 (60 Minutes)

Question – Answer Book

Instructions

1. Attempt **ALL** questions. Write your answers in the spaces provided in this Question - Answer Book.
2. A graphic display calculator is needed.
3. You are suggested to prepare a formula booklet of Applications and Interpretation for IBDP Mathematics when attempting the questions.
4. Supplementary answer sheets and graph papers will be supplied on request.
5. Unless otherwise specified, **ALL** working must be clearly shown.
6. Unless otherwise specified, numerical answers should be either **EXACT** or correct to **3 SIGNIFICANT FIGURES**.
7. The diagrams in this paper are **NOT** necessarily drawn to scale.
8. Information to be read before you start the exam:

	Marker's Use Only	**Examiner's Use Only**	
Question Number	**Marks**	**Marks**	**Maximum Mark**
1			**29**
2			**26**
Overall			
Paper 3 Total			**55**

1. This question aims at investigating the performance of three athletes in a 100 metres running practice.

In January 2021, Patty, Quinn and Rachel are three athletes belonging to the same running group. It is given that the time for Patty, Quinn and Rachel to finish a 100 metres run are normally distributed, with means and standard deviations as follows:

Athlete	Patty	Quinn	Rachel
Variable	P	Q	R
Mean	14.1 s	14.9 s	$\mu\text{ s}$
Standard deviation	0.7 s	0.55 s	$\sigma\text{ s}$

Assume that the time for each of them to finish a 100 metres run are independent of each other.

On one training day, Patty practiced 100 metres run for three times, and the times used are recorded.

(a) Write down, for the sum of the three observations,

(i) the mean;

(ii) the variance.

[2]

(b) Hence, find the probability that Patty used less than 40.5 seconds in total.

[2]

On another training day, Patty and Quinn both practiced 100 metres run for five times, and the times used are recorded.

(c) Find the variance for the mean of

(i) Patty's trials;

(ii) Quinn's trials.

[4]

(d) Hence, find the probability that the mean time for Patty's five trials is

(i) greater than that for Quinn's five trials;

(ii) 0.2 seconds within that for Quinn's five trials.

[8]

On the same day, in order to find the value of μ, Rachel also practiced 100 metres run for five times, and the times used are recorded as follows:

$$13.9\text{ s},\ 14.7\text{ s},\ 13.5\text{ s},\ 14.0\text{ s},\ 14.2\text{ s}$$

(e) Find the unbiased estimates for

(i) μ;

(ii) σ.

[4]

(f) Construct a 95% confidence interval for μ, giving the answer to three decimal places.

[2]

In February 2021, the manager of the running club investigates the best records for six athletes in the club to finish 100 metres runs in January and February:

Athlete	Patty	Quinn	Rachel	Sally	Tina	Ursula
Best record in January 2021 (j s)	13.2 s	13.8 s	13.5 s	15.0 s	14.0 s	13.0 s
Best record in February 2021 (f s)	13.1 s	14.2 s	13.4 s	14.8 s	14.9 s	13.0 s

The manager wants to investigate whether on average the athletes show improvement in February. A paired t-test is conducted at a 5% significance level. Define $d = j - f$.

(g) (i) Write down the null hypothesis of the test.

(ii) Write down the alternative hypothesis of the test.

(iii) Find the p-value.

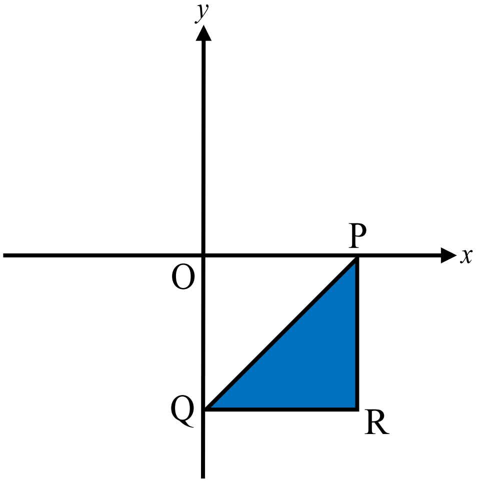
(iv) Write down the value of t, the test statistic.

(v) State the conclusion of the test with a reason.

[7]

2. This question aims at investigating the changes of plane figures in a sequence of slides.

Alfred, Benny and Calix are creating presentation slides showing the sequence of the changes of a plane figure. On the first slide, an isosceles triangle PQR is constructed on a coordinate plane such that $\text{PR} \perp \text{QR}$, and the coordinates of P and Q are $(10, 0)$ and $(0, -10)$ respectively.



(a) Write down

(i) the coordinates of R ;

(ii) the area of the triangle PQR .

[2]

Alfred and Benny plan to generate the next slide by applying the transformation $\begin{pmatrix} x_n \\ y_n \end{pmatrix} = A_n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ to the current slide, where $n \geq 1$ is the number of slides after the first slide, (x_0, y_0) is a point on the triangle PQR where (x_n, y_n) is its image after n transformations, A_n is a 2×2 matrix representing an enlargement about the origin, with a variable scale factor $k(n) = a \cdot b^n$, a, $b \in \mathbb{R}$.

Alfred suggests $a = 2$ and $b = 1$. Let (x_0, y_0) be the coordinates of R .

(b) (i) Find $\begin{pmatrix} x_3 \\ y_3 \end{pmatrix}$.

(ii) Find the area of the triangle $\text{P}_3\text{Q}_3\text{R}_3$, the image of PQR after 3 transformations.

(iii) Show that $x_n - y_n = 5 \cdot 2^{n+2}$.

[11]

Benny suggests $a = 1$ and $b = 1.1$. Let (x_0, y_0) be the coordinates of Q.

(c) (i) Find $\begin{pmatrix} x_4 \\ y_4 \end{pmatrix}$.

(ii) Find the least value of n such that $y_n < -375$.

[10]

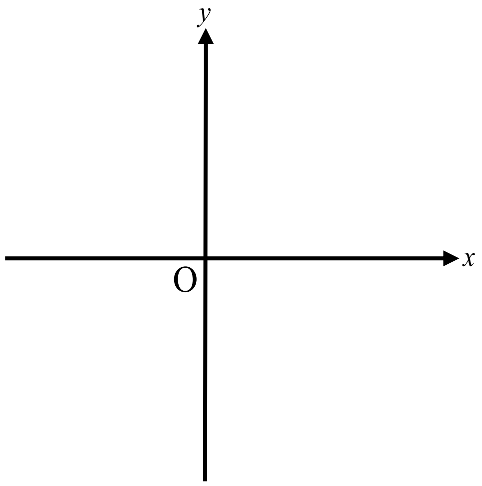
Calix plans to generate the next slide by applying the transformation $\begin{pmatrix} x_n \\ y_n \end{pmatrix} = B_n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ to the current slide, where $n \geq 1$ is the number of slides after the first slide, (x_0, y_0) is a point on the triangle PQR where (x_n, y_n) is its image after n transformations, B_n is a 2×2 matrix representing a reflection about the line $y = (\tan(n \cdot 45^\circ))x$.

Note: The line of reflection is defined as $x = 0$ when n is even.

(d) (i) Express B_2 as the product of two matrices.

(ii) On the following diagram, sketch the triangle $\mathrm{P_2Q_2R_2}$, the image of the triangle PQR after two transformations.

[3]



END OF PAPER

Applications and Interpretation Higher Level for IBDP Mathematics Practice Paper Set 4 – Paper 1 (120 Minutes)

Question – Answer Book

Instructions

1. Attempt **ALL** questions. Write your answers in the spaces provided in this Question - Answer Book.
2. A graphic display calculator is needed.
3. You are suggested to prepare a formula booklet of Applications and Interpretation for IBDP Mathematics when attempting the questions.
4. Supplementary answer sheets and graph papers will be supplied on request.
5. Unless otherwise specified, **ALL** working must be clearly shown.
6. Unless otherwise specified, numerical answers should be either **EXACT** or correct to **3 SIGNIFICANT FIGURES**.
7. The diagrams in this paper are **NOT** necessarily drawn to scale.
8. Information to be read before you start the exam:

	Marker's Use Only	**Examiner's Use Only**	
Question Number	**Marks**	**Marks**	**Maximum Mark**
1			**6**
2			**4**
3			**6**
4			**6**
5			**6**
6			**6**
7			**6**
8			**6**
9			**8**
10			**5**
11			**6**
12			**7**
13			**5**
14			**6**
15			**7**
16			**6**
17			**7**
18			**7**
Overall			
Paper 1 Total			**110**

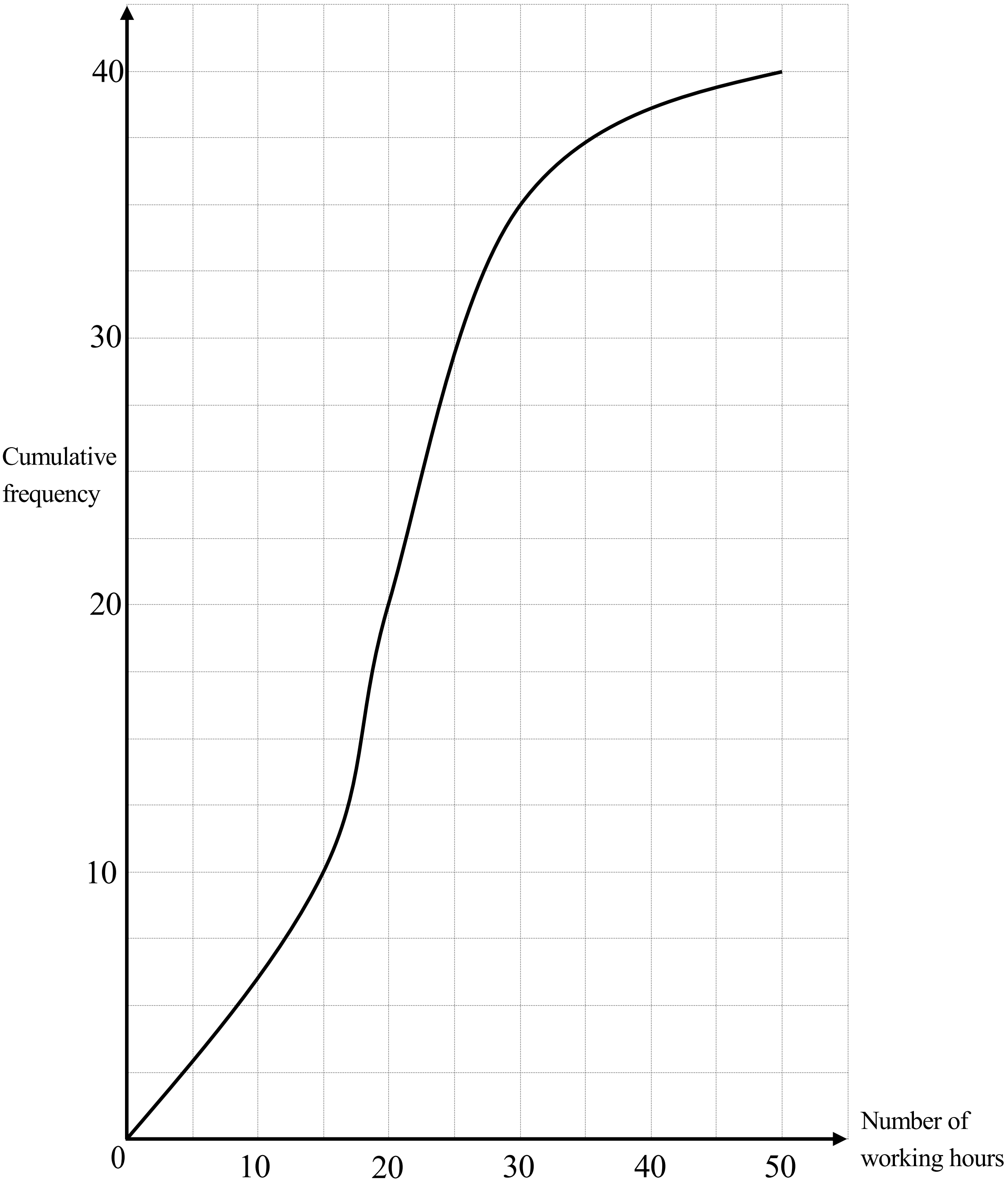
1. A pencil sharpener in the shape of a right circular cone has volume $128\pi\ \text{cm}^3$ and vertical height $6\ \text{cm}$.

(a) Find the radius of the circular base of the pencil sharpener. [2]

(b) Find the slant height of the pencil sharpener. [2]

(c) Hence, find the total surface area of the pencil sharpener. [2]

2. There are 40 workers in a factory. The following cumulative frequency curve shows the number of hours workers worked in a particular week.



(a) Write down

(i) the median number of hours worked by workers;

(ii) the lower quartile of the number of hours worked by workers.

[2]

12.5% of the workers worked for more than k hours in that week.

(b) Find k.

[2]

3. Consider the following four sequences, where only one of them is arithmetic and only one of them is geometric:

$$a_n = 10, 9, 7, 4, \cdots$$
$$b_n = 10, 12.5, 15.625, 19.53125, \cdots$$
$$c_n = 10, 8.9, 7.8, 6.7, \cdots$$
$$d_n = 10, 12, 14, 18, \cdots$$

(a) State which sequence is

(i) arithmetic;

(ii) geometric.

[2]

(b) For the geometric sequence selected in (a)(ii),

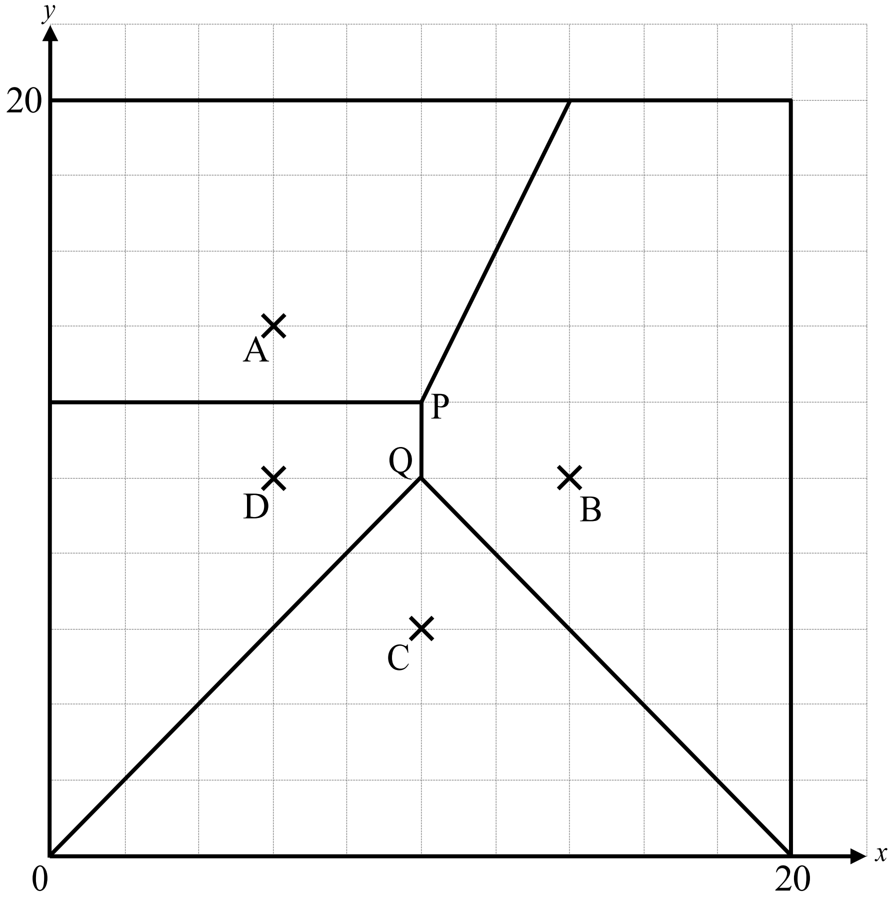
(i) write down the common ratio;

(ii) write down the fifth term, giving the answer in fraction;

(iii) find the sum of the first eight terms.

[4]

4. The diagram below shows the Voronoi diagram of four landfills, A, B, C and D, in a city bounded by the coordinate axes, the lines $x = 20$ and $y = 20$, where 1 unit on the grid represents $2\,\mathrm{km}$. The points P and Q are the intersections of the boundaries of Voronoi cells.



There are two apartments, one is located at P and the another one is located at Q. Jack is going to rent a flat in one of the apartments, and he hopes that the flat selected to be as far as possible from all other landfills.

Two circles are constructed such that the centres are P and Q respectively, and passes through the landfills in adjacent Voronoi cells.

(a) (i) Find the radius of the circle centred at P.

(ii) Write down the radius of the circle centred at Q.

(iii) Hence, determine which apartment is farthest from all four landfills.

[4]

(b) Write down the equation of the boundary separating the Voronoi cells of B and C, giving the answer in the form $Ax + By + C = 0$.

[2]

5. A random variable X follows a binomial distribution such that $X \sim \mathrm{B}(n, 0.72)$, where n is a positive integer. It is given that the expected value of X is 8.64.

(a) Find the value of n.

[2]

(b) Find the value of $\mathrm{Var}(X)$.

[2]

(c) Find the value of $\mathrm{P}(X \geq 11)$.

[2]

6. Mike wants to accumulate an amount of money in an investment plans. He deposits $\$200$ at the end of each month. The interest is earned 4.5% per year.

(a) Let P be the value of the investment after ten years. Find the value of P.

[3]

(b) If Mike wants to adjust the amount of deposit such that the value of the investment after twelve years is $5P$, find the new amount of deposit.

[3]

7. A bullet is shot from a submarine which is stationary on the sea level. The path of the bullet above the sea level can be modelled by the function $V = -x^2 + 100x - 1600$, where V m represents the height of the bullet above the sea level, and x m represents the horizontal distance of the bullet from an observer.

(a) Find x when V attains its maximum.

[2]

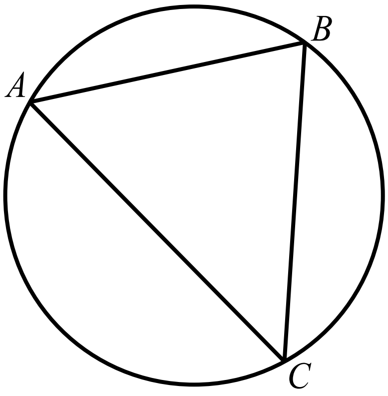
(b) Hence, show that the maximum height of the bullet above the sea level is 900 m.

[1]

(c) Find the **horizontal** distance of the bullet travelled above the sea level.

[3]

8. The triangle ABC is drawn in a circle such that its vertices are on the circumference of the circle.



$\mathrm{AB} = 13.9\,\mathrm{cm}$, $\mathrm{AC} = 17.7\,\mathrm{cm}$ and $\mathrm{A\hat{B}C} = 60.8^{\circ}$

(a) Find the size of the angle $\mathrm{A\hat{C}B}$.

[3]

(b) Hence, find the area of the triangle ABC.

[3]

9. Consider the differential equation $\frac{\mathrm{d}x}{\mathrm{d}t} = \pi x^2 \cos \pi t$, where t, $x > 0$.

(a) Express $\int \frac{1}{x^2} \mathrm{d}x$ as an indefinite integral of t.

[2]

(b) Hence, using the substitution $u = \pi t$ to find the expression of $\frac{1}{x}$, giving the answer in terms of t.

[3]

It is given that $x = 1$ when $t = 2.5$.

(c) Find the expression of x.

[3]

10. The Body Mass Index (BMI) of a group of tennis players is studied. Ten players are randomly selected and their BMIs are measured. It is given that the 95% confidence interval for the population mean is $(18.95, 25.15)$.

(a) Find an unbiased estimate of the population mean.

[2]

Let σ^2 be the known population variance.

(b) Find σ.

[3]

11. Let $f(x)=\sqrt{3-x}$, for $x \le 3$.

(a) Find $f^{-1}(10)$.

[3]

(b) Let g be a function such that $g(5)=\pi$ and g^{-1} exists for all real numbers.

(i) Write down $g^{-1}(\pi)$.

(ii) Hence, find $(f^{-1} \circ g^{-1})(\pi)$.

[3]

12. The following table shows the age of a car x (in years) and the corresponding number of oil refill.

Age of a car $(x \text{ years})$	1	2	3	4	5
Number of oil refill (y)	45	82	135	150	171

(a) The relationship between the variables is modelled by the regression equation $y = ax + b$.

(i) Write down the value of a and of b.

(ii) Hence, estimate the number of oil refills of a car when its age is 2.5 years.

[4]

(b) (i) Write down the correlation coefficient.

(ii) Write down the coefficient of determination.

(iii) Hence, interpret the coefficient of determination.

[3]

13. The following weight adjacency matrix shows the information of a graph with five vertices A, B, C, D and E:

	A	B	C	D	E
A	-	30	20	10	32
B	30	-	25	28	22
C	20	25	-	26	36
D	10	28	26	-	24
E	32	22	36	24	-

Prim's algorithm, starting at E, is used to find the minimum spanning tree for this graph.

(a) State the edge of the greatest weight.

[1]

(b) By using the algorithm, find the minimum spanning tree.

[3]

(c) Write down the weight of the minimum spanning tree.

[1]

14. A die is tossed for 150 times. From the observation, 39 results are shown as '3's.

A hypothesis test is conducted at a 5% significance level to test whether the die is biased such that the probability of getting a '3' is greater than 0.25.

(a) (i) Write down the null hypothesis of the test.

(ii) Write down the alternative hypothesis of the test.

[2]

(b) Find the p-value.

[2]

(c) State the conclusion of the test with a reason.

[2]

15. Let $f(x) = e^{5x}$.

(a) Find $f^{-1}(x)$.

[3]

(b) Write down the range of f^{-1}.

[1]

Let $g(x) = (3 + \ln x)^2$.

(c) Express $(g \circ f)(x)$ in the form $ax^2 + bx + c$, where a, b and c are integers.

[3]

16. Let $\mathbf{T} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$.

(a) Describe the transformation represented by $\mathbf{T}$.

[1]

(b) $\mathbf{T}$ transforms the point P to the point $(8, 0)$. Find the coordinates of P.

[3]

(c) Write down the smallest positive integer n such that $\mathbf{T}^n = \mathbf{I}$, where $\mathbf{I}$ is a 2×2 identity matrix.

[2]

17. Let $f(x)=\ln(x^2+4)$.

(a) Find $f'(x)$.

[2]

(b) Hence, write down the gradient of the tangent to the curve of f at $(3, \ln 13)$.

[1]

It is given that the equation of the normal to the curve of f at $(3, \ln 13)$ is $13x+my=39+m\ln 13$, where m is a nonzero constant.

(c) Find the exact value of the x-intercept of the normal to the curve of f at $(3, \ln 13)$.

[4]

18. The matrix $\mathbf{T} = \begin{pmatrix} 0.73 & 0.31 \\ 0.27 & 0.69 \end{pmatrix}$ is a transition matrix for a Markov chain.

(a) Find the values of λ_1 and λ_2, the eigenvalues of $\mathbf{T}$, where $\lambda_1 < \lambda_2$.

[3]

Let $\mathbf{v}_0 = \begin{pmatrix} 0.4 \\ 0.6 \end{pmatrix}$.

(b) Find $\mathbf{v}_{10}$, the state probability vector after ten transitions.

[2]

(c) Find $\mathbf{v}$, the steady state probability vector for this Markov chain.

[2]

END OF PAPER

Applications and Interpretation Higher Level for IBDP Mathematics
Practice Paper Set 4 – Paper 2 (120 Minutes)

Question – Answer Book

Instructions

1. Attempt **ALL** questions. Write your answers in the spaces provided in this Question - Answer Book.
2. A graphic display calculator is needed.
3. You are suggested to prepare a formula booklet of Applications and Interpretation for IBDP Mathematics when attempting the questions.
4. Supplementary answer sheets and graph papers will be supplied on request.
5. Unless otherwise specified, **ALL** working must be clearly shown.
6. Unless otherwise specified, numerical answers should be either **EXACT** or correct to **3 SIGNIFICANT FIGURES**.
7. The diagrams in this paper are **NOT** necessarily drawn to scale.
8. Information to be read before you start the exam:

	Marker's Use Only	**Examiner's Use Only**	
Question Number	**Marks**	**Marks**	**Maximum Mark**
1			**20**
2			**16**
3			**19**
4			**14**
5			**14**
6			**14**
7			**13**
Overall			
Paper 2 Total			**110**

1. The straight line L_1 passes through the points $\mathrm{A}(30, 0)$ and $\mathrm{B}(0, 40)$.

(a) Find the gradient of L_1.

[2]

(b) Find the equation of L_1, giving the answer in the form $Ax + By + C = 0$.

[2]

Another straight line L_2 passes through the origin O and is perpendicular to L_1.

(c) Find the equation of L_2, giving the answer in the form $y = mx + c$.

[2]

L_1 and L_2 intersects at the point C.

(d) Find the coordinates of C.

[3]

(e) Find the area of the triangle OBC.

[2]

(f) Find the perimeter of the triangle OBC.

[4]

The point D is on L_2 such that k is the x-coordinate of D, where $k < 0$.

(g) Write down the y-coordinate of D, giving the answer in terms of k.

[1]

It is given that the area of the triangle BCD is 624.

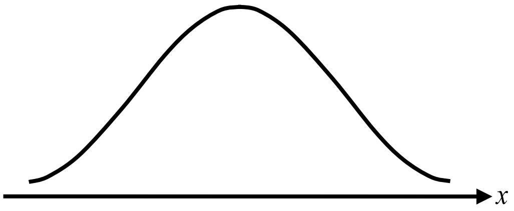
(h) Find k.

[4]

2. A company produces containers of milk soda. The volume of milk soda in the containers is normally distributed with a mean of 500 ml and standard deviation of 8 ml. A container which contains less than 490 ml of milk soda is unsatisfied.

(a) On the following diagram, shade the region representing the probability that a container is unsatisfied.

[2]



(b) A container is selected at random.

(i) Find the probability that this container is unsatisfied.

(ii) Given that this container is unsatisfied, find the probability that it contains more than 483 ml of milk soda.

[5]

(c) Two containers are selected at random. Find the probability that only one of them is unsatisfied.

[3]

Ethan randomly bought ten containers from a department store. He has an agreement with his teacher that the teacher needs to pay Ethan $\$10$ if none of the containers is unsatisfied; Ethan needs to pay his teacher $\$5$ if one or two containers are unsatisfied, and Ethan needs to pay his teacher $\$20$ if more than two containers are unsatisfied.

The above agreement can be summarized in the following table:

The amount that Ethan receives from his teacher	$-\$20$	$-\$5$	$\$10$
Probability	q	$1-p-q$	p

(d) Write down the value of

(i) p;

(ii) q;

(iii) the expected amount that Ethan receives from his teacher.

[6]

3. The function f is defined as $f(x) = -x^3 + 13x^2 - 40x + 36$ for $x \geq 2$.

(a) Write down

(i) the coordinates of the maximum point of $f(x)$;

(ii) the range of values of x such that $f(x)$ is increasing.

[4]

(b) (i) Find $f'(x)$.

(ii) Write down $f'(5)$.

(iii) Hence, show that the equation of the tangent to the graph of f at $x = 5$ is $15x - y - 39 = 0$.

[6]

$f(x) \geq 0$ for $2 \leq x \leq p$. Let R be the region enclosed by the graph of f and the x-axis.

(c) (i) Write down the value of p.

(ii) Write down the integral representing the area of R.

(iii) Hence, find the exact value of the area of R.

[4]

(d) Use the trapezoidal rule with 4 intervals, find an estimate of the area of R.

[4]

(e) State whether the estimate in (d) overestimates or underestimates $\int_0^8 f(x)\mathrm{d}x$.

[1]

4. A ball A is moving on a smooth table which can be modelled by a x-y plane. Its initial position is $(0, 0)$. A collides with another moving ball B after two seconds. It is given that the initial position of B is $(12, 5)$.

(a) Find the initial distance between the two balls.

[2]

The velocity vector of A is given by $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$.

(b) Write down the vector equation for the displacement of A, giving the answer in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, where t is the time in seconds after the start of motion.

[2]

B stops moving after colliding with A.

(c) Find the velocity vector of B.

[2]

A changes its velocity to $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ after the collision. It is given that A is at $(5.5, 17.5)$ x seconds after collision.

(d) Find x.

[2]

(e) Find the acute angle between the two velocity vectors of A.

[3]

It is given that the horizontal edge of the table is given by the equation $y = 31$.

(f) Find the amount of time needed for A to reach the edge after it starts its motion.

[3]

5. Let x and y be the populations, in thousands, of brown bears and giant pandas in a national park respectively. The changes in the populations can be modelled by the coupled differential equations $\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t}+6x=0 \\ \dfrac{\mathrm{d}y}{\mathrm{d}t}-5y=-x \end{cases}$.

(a) When $x=5$, state the range of values of y such that $\dfrac{\mathrm{d}y}{\mathrm{d}t}<25$.

[1]

The system can be expressed by a matrix equation $\dot{\mathbf{X}}=\mathbf{MX}$, where $\mathbf{M}$ is a 2×2 matrix, and $\dot{\mathbf{X}}=\begin{pmatrix} \dfrac{\mathrm{d}x}{\mathrm{d}t} \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} \end{pmatrix}$ and $\mathbf{X}=\begin{pmatrix} x \\ y \end{pmatrix}$ are two 2×1 matrices. Let λ_1 and λ_2 be the eigenvalues of $\mathbf{M}$, where $\lambda_1<\lambda_2$.

(b) Find $\det(\mathbf{M}-\lambda\mathbf{I})$, giving the answer in terms of λ.

[2]

(c) Hence, write down the values of λ_1 and λ_2.

[2]

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the eigenvectors of $\mathbf{M}$ corresponding to λ_1 and λ_2 respectively.

(d) Write down $\mathbf{v}_1$ and $\mathbf{v}_2$.

[2]

The initial populations of brown bear and giant panda are 22000 and 5000 respectively.

(e) Find the particular solution of

(i) x;

(ii) y.

[5]

(f) Hence, state the long-term behaviour for the population of

(i) brown bear;

(ii) giant panda.

[2]

6. Two alternating current electrical sources are given as $V_1 = 23\sin(6\pi t - 0.17)$ and $V_2 = \mathrm{Im}(e^{6\pi t \mathrm{i}}(z-w))$ respectively, where t represents time in seconds, and z, $w \in \mathbb{C}$. The total voltage V is given by $V = V_1 + V_2 = 29\sin(6\pi t - 0.31)$.

(a) Find the expression of $z-w$, where z is in the form $re^{-0.31\mathrm{i}}$.

[3]

(b) Express the following in the form $r(\cos\theta + i\sin\theta)$:

(i) z

(ii) w

[2]

(c) It is given that $z-w = Le^{\mathrm{i}\alpha}$. Find

(i) L;

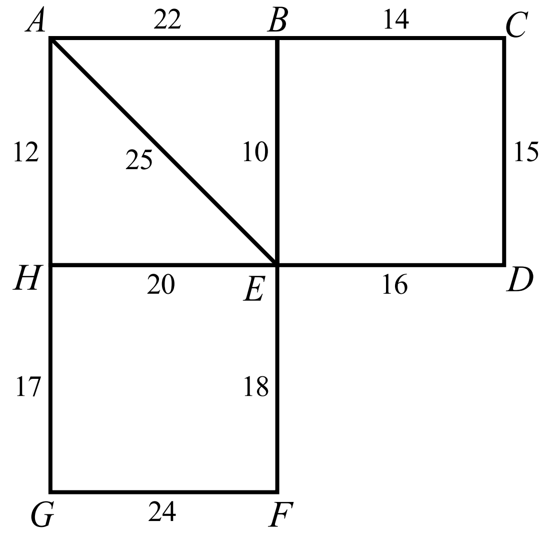
(ii) α.

[6]

(d) Hence, express V_2 in the form $A\sin(Bt+C)$, A, B, $C \in \mathbb{R}$.

[3]

7. The following weighted graph shows a network of railways connecting eight train stations A, B, C, D, E, F, G and H, in a town. The weight on each edge shows the transportation fee (in dollars) for a passenger to travel between two adjacent stations by train.



(a) Write down

(i) the degree of E;

(ii) the number of vertices of odd degree;

(iii) the number of vertices of even degree;

(iv) the minimum amount of transportation fee needed to travel from A to F.

(v) the minimum amount of transportation fee needed to travel from C to G.

[5]

(b) Use the Chinese postman algorithm to find a possible route of minimum transportation fee required to pass through all edges, starting at H and finishing at B.

[3]

(c) Write down the corresponding transportation fee required of the route.

[1]

Assume that it is necessary to start and finish at the same point. Joseph starts his journey at the station B.

(d) (i) Use the Chinese postman algorithm to find a possible route of minimum transportation fee required for Joseph to pass through all edges.

(ii) Write down the corresponding transportation fee required of the route.

[4]

END OF PAPER

Applications and Interpretation Higher Level for IBDP Mathematics
Practice Paper Set 4 – Paper 3 (60 Minutes)

Question – Answer Book

Instructions

1. Attempt **ALL** questions. Write your answers in the spaces provided in this Question - Answer Book.

2. A graphic display calculator is needed.

3. You are suggested to prepare a formula booklet of Applications and Interpretation for IBDP Mathematics when attempting the questions.

4. Supplementary answer sheets and graph papers will be supplied on request.

5. Unless otherwise specified, **ALL** working must be clearly shown.

6. Unless otherwise specified, numerical answers should be either **EXACT** or correct to **3 SIGNIFICANT FIGURES**.

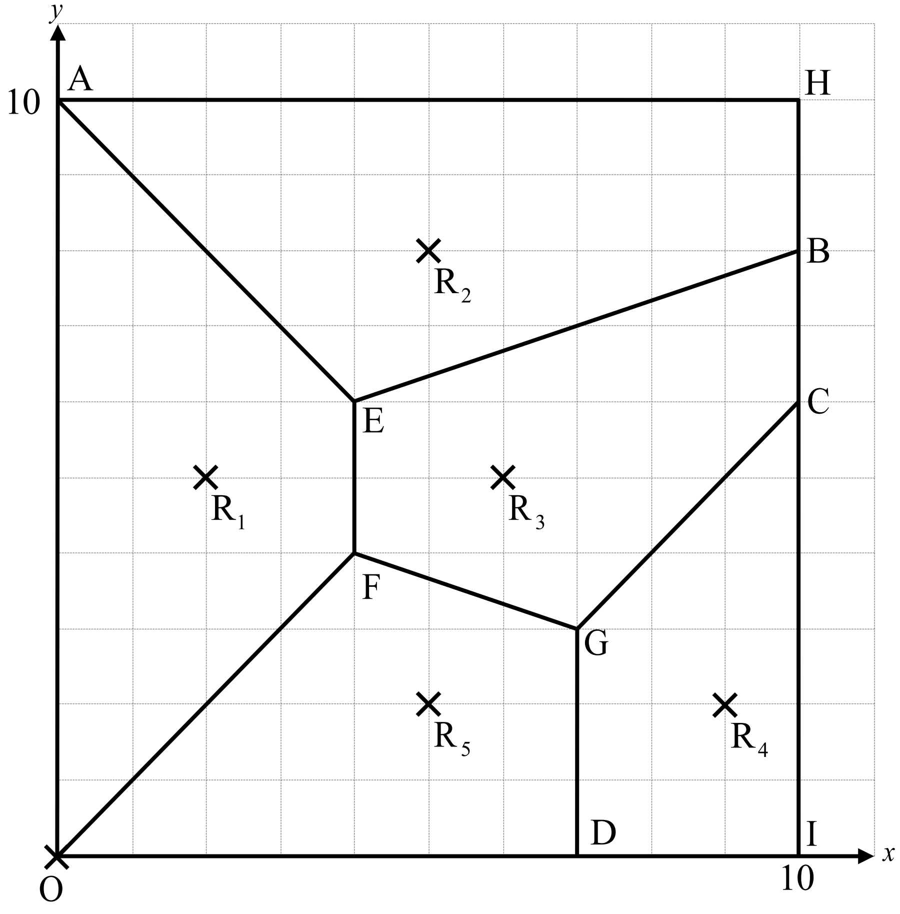
7. The diagrams in this paper are **NOT** necessarily drawn to scale.

8. Information to be read before you start the exam:

	Marker's Use Only	**Examiner's Use Only**	
Question Number	**Marks**	**Marks**	**Maximum Mark**
1			**30**
2			**25**
Overall			
Paper 3 Total			**55**

1. This question aims at investigating the road traffic conditions by Voronoi diagrams and graph theory.

The diagram below shows the Voronoi diagram of five reservoirs, R_1, R_2, R_3, R_4 and R_5, in a country side of dimension $10\text{ km}\times10\text{ km}$, where 1 unit on the coordinate plane represents 1 km.



Initially, only four roads of 10 km each are built along OA, AH, HI and IO. Later, seven more roads are built along the boundaries of Voronoi cells, where the sites are all reservoirs. O, A, B, C, D, E, F, G, H and I are towns located on the intersections of the roads.

(a) (i) Write down the area of the Voronoi cell of the reservoir R_2.

(ii) Write down the gradient of the road separating the Voronoi cells of the reservoirs R_2 and R_3.

(iii) Hence, find its equation.

(iv) State the significance of the Voronoi cell of the reservoir R_3.

[5]

(b) If one extra reservoir is added to $(3, 1)$ such that the above road plan in the Voronoi diagram is modified, write down the road which is affected and has to be reconstructed.

[1]

This Voronoi diagram can also be considered as a graph of ten vertices, where the vertices represent the towns O, A, B, C, D, E, F, G, H and I, and the roads connecting the towns are considered as edges.

(c) Write down

(i) the number of edges;

(ii) the number of vertices of odd degree;

(iii) the number of vertices of even degree;

(iv) the adjacency matrix **M** of the graph.

[8]

(d) Hence, write down the total number of walks of length at least four and at most six from A to B.

[2]

(e) (i) State one possible Hamiltonian cycle.

(ii) State one possible Hamiltonian path that starts at H and end at G.

(iii) Explain why an Eulerian circuit does not exist.

[5]

The following table shows some of the entries of the least distance of a path, in kilometres, connecting any two towns. The distances are calculated and correct to three significant figures if necessary.

	O	A	B	C	D	E	F	G	H	I
O	-	10	*	*	7	*	5.66	q	*	*
A	10	-	*	*	*	5.66	p	*	10	*
B	*	*	-	2	*	6.32	*	*	2	*
C	*	*	2	-	*	*	*	4.24	*	6
D	7	*	*	*	-	*	*	3	*	3
E	*	5.66	6.32	*	*	-	2	*	*	*
F	5.66	p	*	*	*	2	-	3.16	*	*
G	q	*	*	4.24	3	*	3.16	-	*	*
H	*	10	2	*	*	*	*	*	-	*
I	*	*	*	6	3	*	*	*	*	-

(f) Write down the value of

(i) p;

(ii) q.

[2]

Due to heavy snow, roads AH, HB, BC, CI, EB, GC and DI are blocked such that vehicles cannot use the above roads.

The following modified table shows the least distance of a path, in kilometres, connecting any two towns from O, A, D, E, F and G.

	O	A	D	E	F	G
O	-	10	7	7.66	5.66	q
A	10	-	13.8	5.66	p	10.8
D	7	13.8	-	8.16	6.16	3
E	7.66	5.66	8.16	-	2	5.16
F	5.66	p	6.16	2	-	3.16
G	q	10.8	3	5.16	3.16	-

(g) Using the nearest neighbour algorithm, starting and finishing at A, find an upper bound of the total distance of a cycle that passes through all six towns, giving the answer correct to one decimal place.

[3]

(h) Using the deleted vertex algorithm by deleting the vertex A, find a lower bound of the total distance of a cycle that passes through all six towns, giving the answer correct to one decimal place.

[4]

2. This question aims at investigating the weights of apples and oranges in a food shop.

Apples and oranges are available in a food shop. The weights of apples can be modelled by a normal distribution with mean 100 g and variance 10 g^2, and the weights of oranges can be modelled by a normal distribution with mean $\mu\text{ g}$ and variance 6 g^2.

(a) One apple and two oranges are randomly selected. Let $\mu = 120$.

(i) Write down the mean of the total weight.

(ii) Write down the variance of the total weight.

(iii) Hence, find the probability that the total weight of one apple and two oranges is between 321 g and 337 g.

[4]

(b) Let $\mu = 125$ and D represents the weight of four randomly selected apples subtracted by the triple of the weight of a randomly selected orange. It is given that $D \sim \mathrm{N}(m, k^2)$.

(i) Write down m.

(ii) Write down the exact value of k.

(iii) Hence, find the probability that the weight of four randomly selected apples is less than three times the weight of a randomly selected orange.

[5]

Kylian and Lucy are staffs in the food shop. Kylian wants to conduct a hypothesis test at a 5% significance level to test whether the population mean weight of oranges is less than 120 g.

Kylian selected 20 oranges randomly to form a sample such that the sample mean weight is 119 g.

(c) (i) Write down the null hypothesis of the test.

(ii) Write down the alternative hypothesis of the test.

(iii) Find the p-value.

(iv) State the conclusion of the test with a reason.

[6]

Lucy wants to conduct another hypothesis test at a certain significance level to test whether the population mean weight of oranges is less than 120 g. She chooses rejecting the null hypothesis if the sample mean is less than 119.2 g. The sample size used is 21.

(d) Find the probability that a Type I error is made.

[2]

The actually value of μ is 119.6.

(e) Find the probability that a Type II error is made.

[2]

Later, Lucy takes another random sample of n oranges to calculate a 90% confidence interval for μ. It is given that the upper bound of the confidence interval is $1.6449v$ greater than the sample mean weight, where v is the standard deviation of the sample mean weight.

(f) (i) Express v in terms of n.

(ii) Hence, find the least value of n such that the width of the confidence interval is not greater than 1.1 grams.

[6]

END OF PAPER